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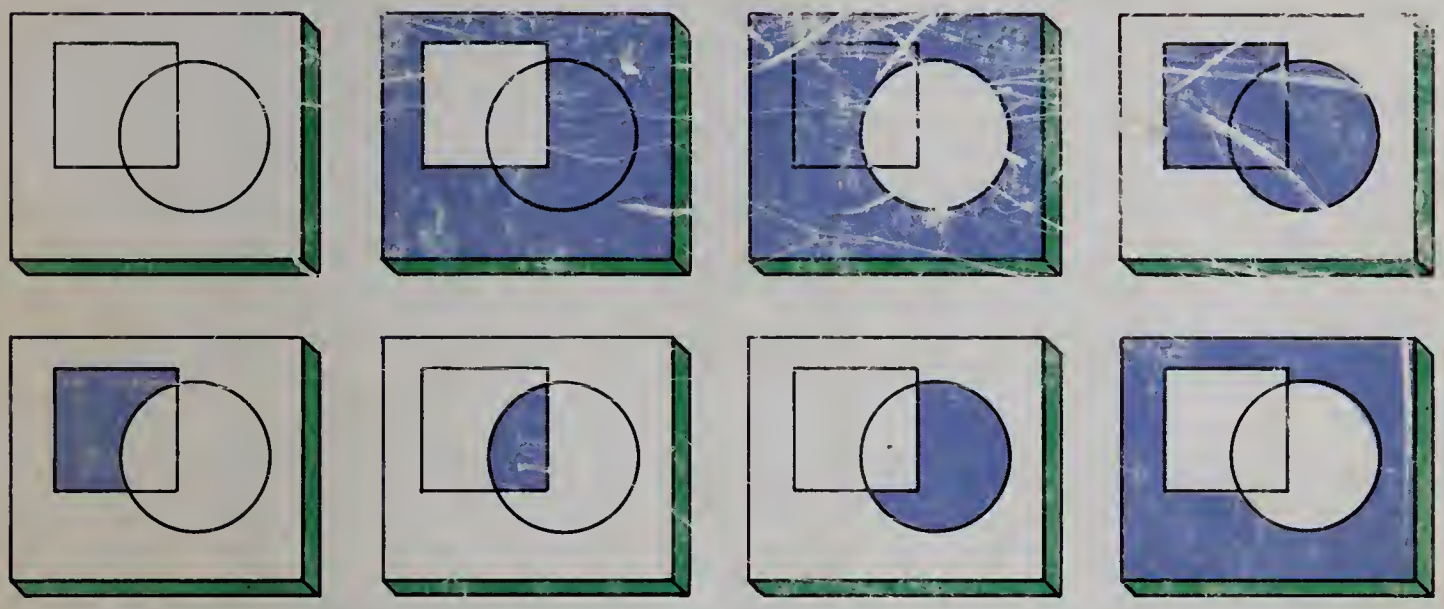
math workshop for children



QA
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lev.F

CURR

ROBERT W. WIRTZ
MORTON BOTEL
W. W. SAWYER

MAT LDREN

Robert W. Wirtz

Assistant Professor of Education, University of Illinois
Staff Research Associate, Addressograph-Multigraph Corporation
Mathematics Consultant, Bucks County Schools, Pennsylvania

Morton Botel

Assistant Superintendent and Reading Consultant, Bucks County Schools, Pennsylvania

W. W. Sawyer

Professor of Mathematics, Wesleyan University, Connecticut

Marvin Levitt

Numo drawings and Art Direction

Art Staff:

Lieselotte Esser, Dorothy Kurr, Roy Lipschutz

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X	.01	.1	1	10	100
.01	.0001	.001	.01	.1	
.1		.01	.1		
1			1		
10				100	
100					10,000

7	11	10		16	20	45
3	6	2	9	5		
4	5	8	7			
12	30	16			100	500

Pattern Hunt

2 x 0 = 1 x 1 – ____

3 x 1 = 2 x 2 – ____

4 x 2 = 3 x 3 – ____

5 x 3 = 4 x 4 – ____

6 x 4 = 5 x 5 – ____

10 x 8 = 9 x ____ – ____

11 x 9 = 10 x ____ – ____

21 x 19 = ____ x ____ – ____

31 x 29 = ____ x ____ – ____

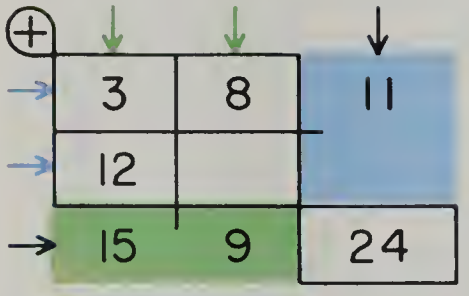
101 x 99 = ____ x ____ – ____

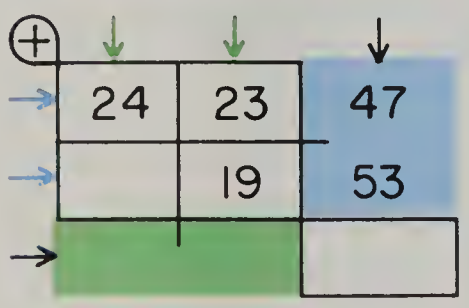
1,001 x 999 = ____ x ____ – ____

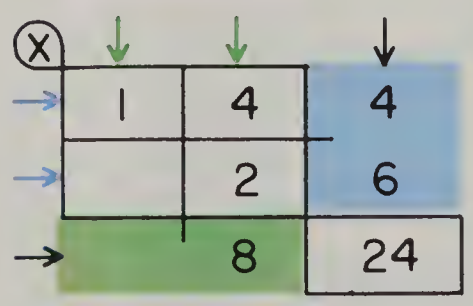
1 1 2 3 5 ___ 13 21 34 ___ ___ ___ ___ 233 ___ ___

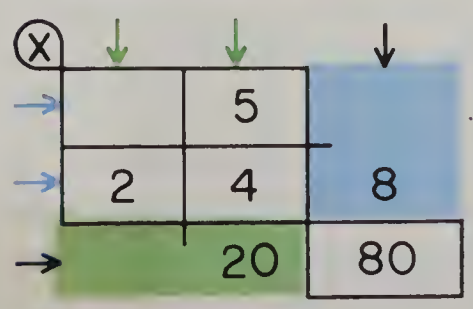
2 1 3 4 7 ___ ___ ___ ___ ___ ___ ___ ___ 521 ___

3 2 5 ___ ___ 19 31 ___ ___ ___ ___ ___ ___ ___ 1,453









2 , 7 → 13

4 , 1 → 3

8 , 3 → 23

5 , 2 → 9

3 , 7 → 20

6 , 5 → ___

2 , 19 → ___

8 , 12 → ___

5 , ___ → 34

___ , 9 → 26

___ , ___ → 76

___ , ___ → 38

8 x 3 + 8 x 4 = 8 x 7 = 56

7 x 6 + 7 x 4 = 7 x 10 = 70

9 x 8 + 9 x 2 = 9 x 10 = 90

4 x 9 + 4 x 1 = 4 x ___ = ___

5 x 13 + 5 x 7 = ___ x ___ = ___

6 x 49 + 6 x 51 = ___ x ___ = ___

13 x 22 + 13 x 8 = ___ x ___ = ___

18 x 3 + 7 x 3 = 25 x ___ = ___

35 x 9 + 5 x 9 = ___ x ___ = ___

87 x 39 + 13 x 39 = ___ x ___ = ___

0 1 8 27 64 ___ ___ ___

1 7 19 ___ ___ ___ ___

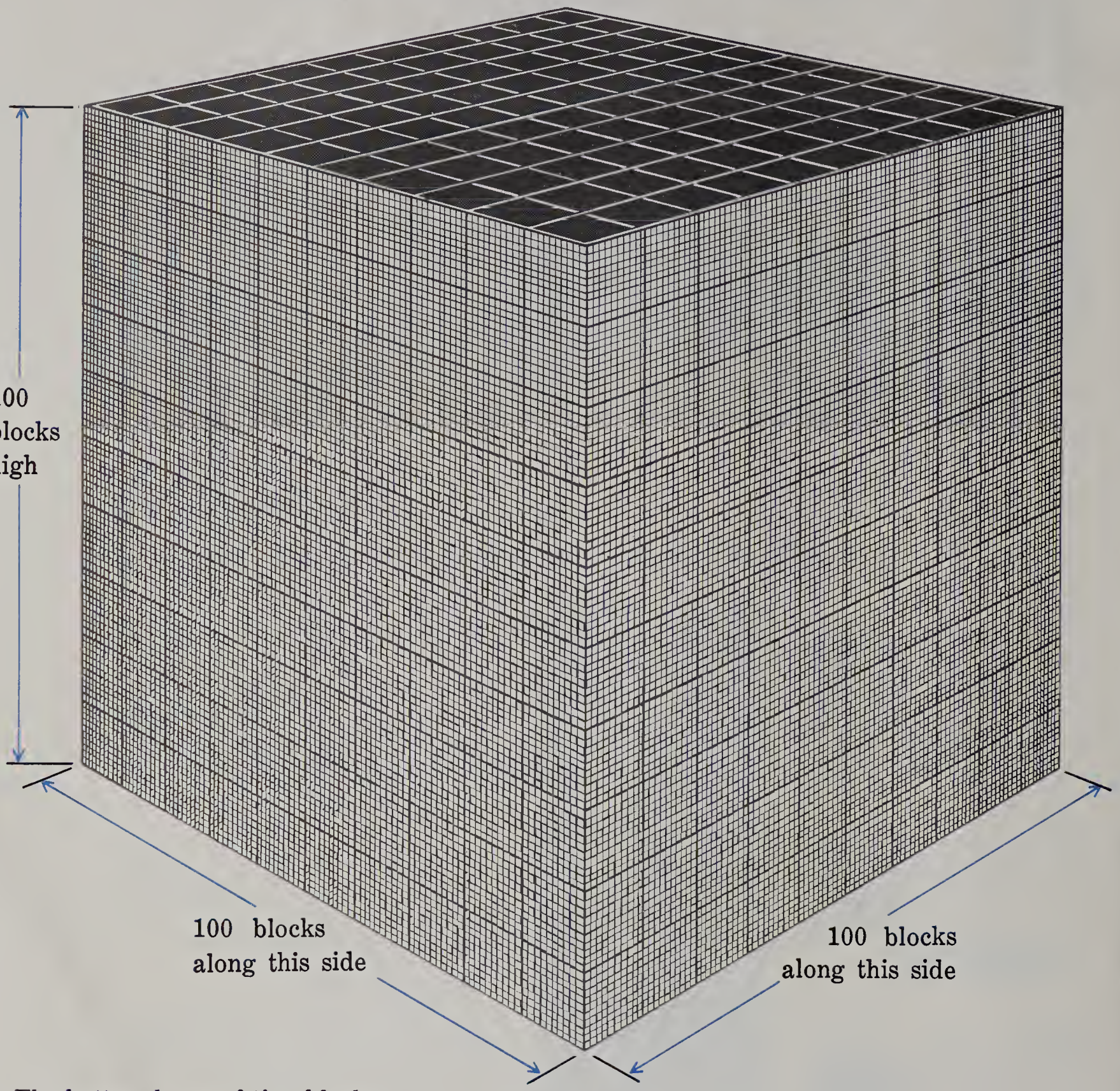
6 12 ___ ___ ___ ___

6 6 6 ___ ___

463155

Can you count to a Million?

You would have to count to a million if you wanted to count all the blocks in a pile as big as the one illustrated in this drawing.



The bottom layer of tiny blocks would have 100 in each of 100 rows.

How much is 100×100? _____

There would be 100 such layers?

How many in 100 layers?

$100 \times 100 \times 100 =$ _____

Of course, you see only a few of the ...s in this pile—those on the outside.

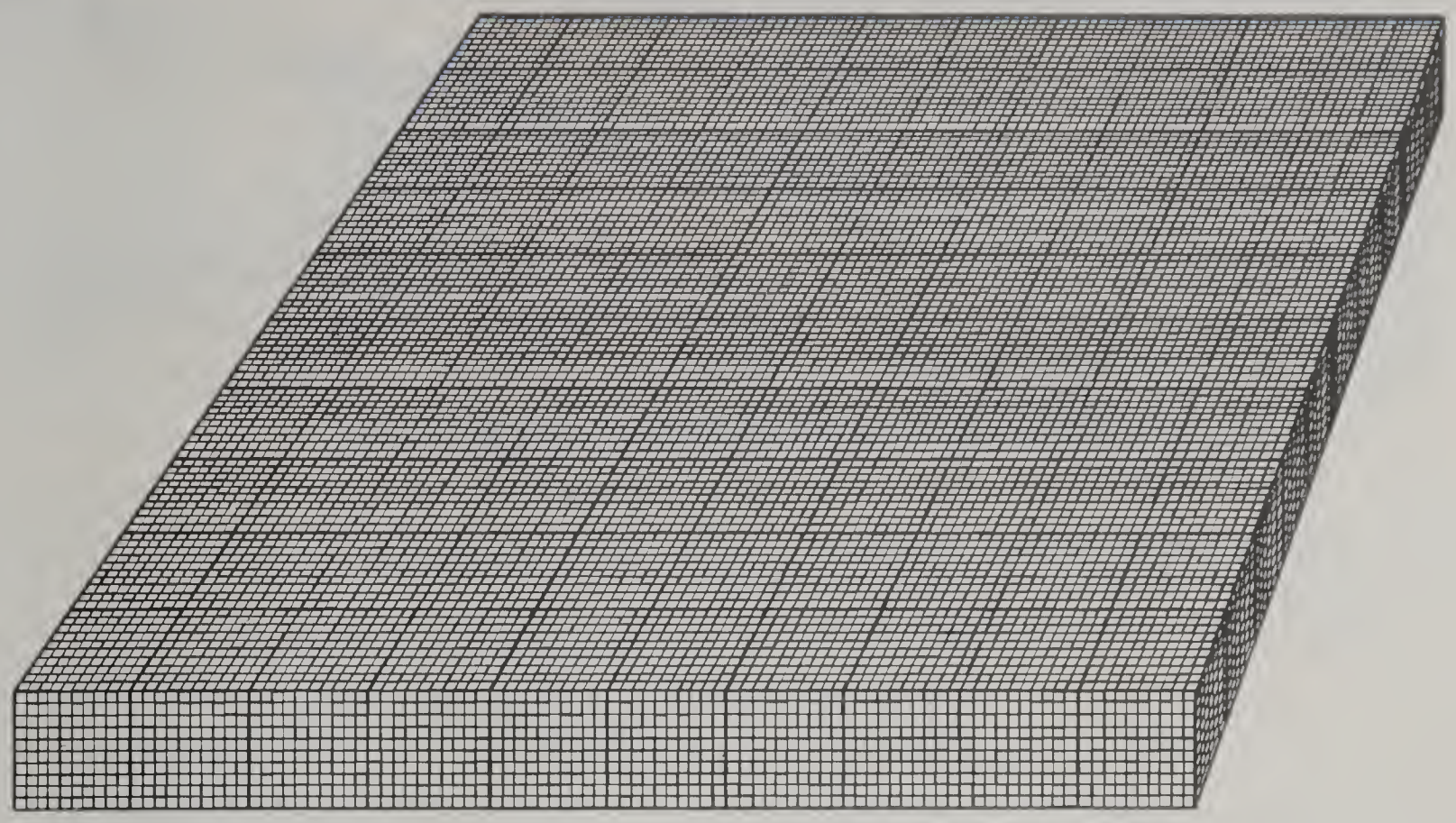
1,000,000 Blocks

. . . one million of them!

Do you think there are a million grains in a bushel of wheat?

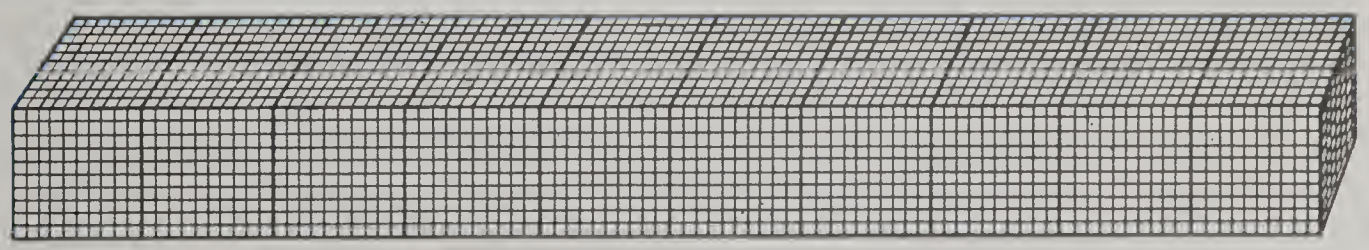
Believe it or not, there are about 5,000,000 grains in one bushel of wheat.

Ten layers of the tiny blocks . . . with 100× 100 in each layer.

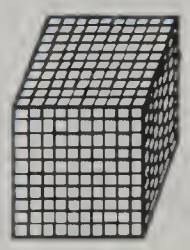


100,000 blocks . . . one hundred thousand of them.

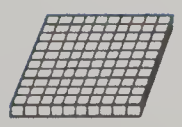
100 × 100 is 10,000! Again, you see only one side of some, two sides of others, and three sides of one. (When you have a little time to spare, you might like to determine how many of the hundred thousand blocks in the pile are showing in the picture.)



10,000 blocks . . . ten thousand blocks



1,000 blocks

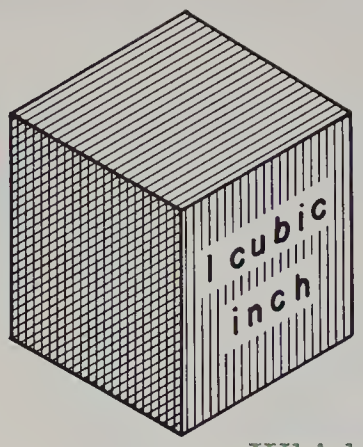


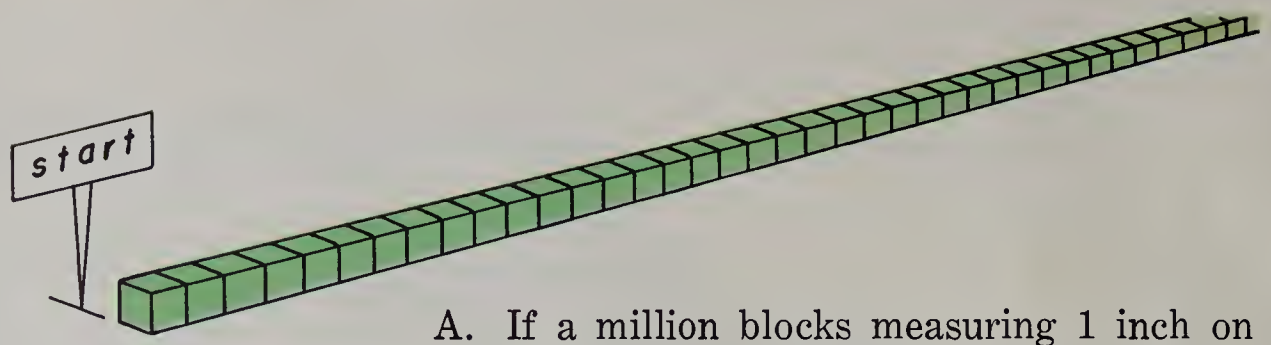
100 blocks

10 blocks

1 block

Each sketch illustrates a pile that has 10 times as many blocks as the sketch that follows it . . . or, starting with the single block at the right, each sketch illustrates $\frac{1}{10}$ of the next larger pile of blocks.





A. If a million blocks measuring 1 inch on all sides — 1 cubic inch — were laid side by side in a straight line, how far would they extend?

Which of the following would be your first guess as the closest to being correct:

500 feet; 5 miles; 15 miles; 100 miles; 770 miles.

B. There are 5,280 feet in a mile.
Can you find out which distance given above is close to the correct answer?

_ _ _ _ _ _ _ _ _ _ _

C. If each one cubic inch block of a certain material weighs 1 ounce, how many tons would a million of those blocks weigh? (There are 16 ounces in a pound and there are 2,000 pounds in a ton.) _ _ _ _ _ _ _ _ _
Which of the weights below is closest to being the correct weight?

1 ton; 19 $\frac{1}{2}$ tons; 31 $\frac{1}{4}$ tons; 87 tons

D. If a million one cubic inch blocks were laid in the pattern below, how far would they reach? _ _ _ _ _ _ _ _ _ _

15 miles; 19 miles; 22 miles; 25 miles; 30 miles

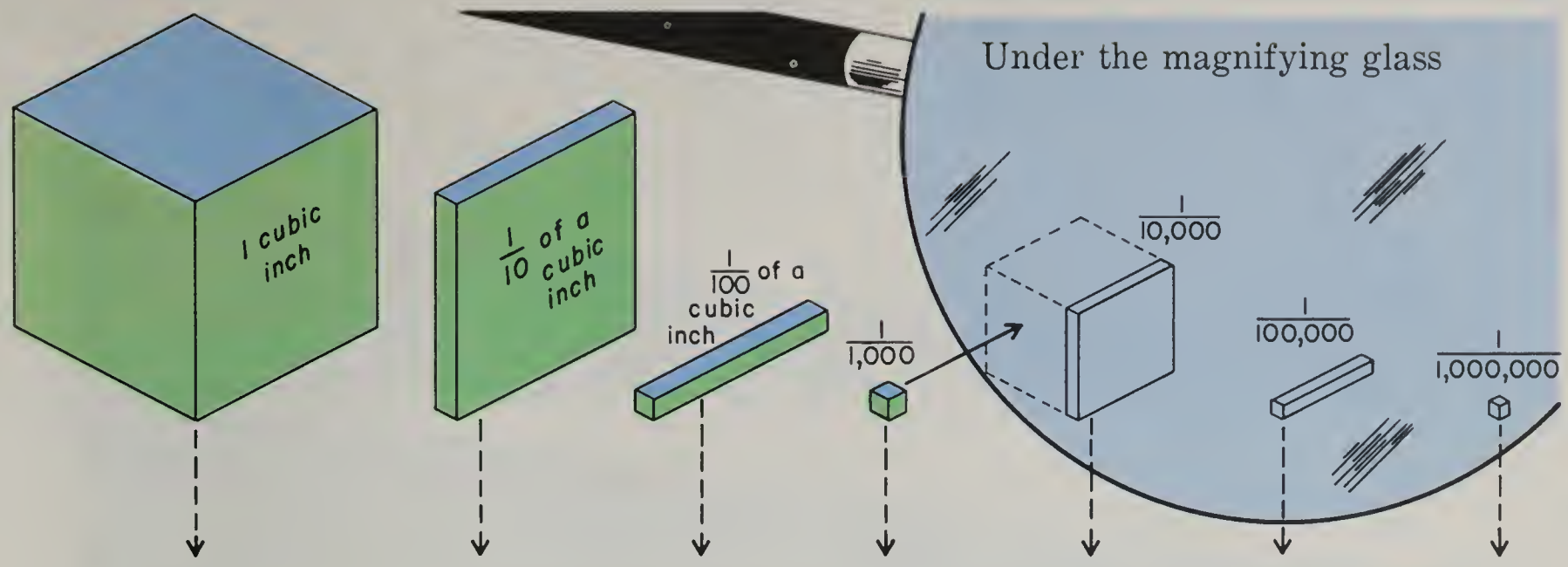
E. How many running feet of 2" × 4" blocks can be cut from 1,000,000 cubic inches of wood?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

F. How many square feet could be covered by 1,000,000 cubes of wood that are 1" by 1" by 1"?

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

1,000,000 ÷ 2 = _____
1,000,000 ÷ 3 = _____
1,000,000 ÷ 4 = _____
1,000,000 ÷ 5 = _____
1,000,000 ÷ 6 = _____
1,000,000 ÷ 7 = _____
1,000,000 ÷ 8 = _____
1,000,000 ÷ 9 = _____
1,000,000 ÷ 10 = _____
1,000,000 ÷ 11 = _____
1,000,000 ÷ 12 = _____
1,000,000 ÷ 13 = _____
1,000,000 ÷ 14 = _____
1,000,000 ÷ 15 = _____
1,000,000 ÷ 20 = _____
1,000,000 ÷ 25 = _____
1,000,000 ÷ 75 = _____
1,000,000 ÷ 100 = _____
1,000,000 ÷ 500 = _____
1,000,000 ÷ 750 = _____
1,000,000 ÷ 1,000 = _____



The weight of each piece above in ounces, if a 1 cubic inch block weighs 1 ounce.

1	.1	.01

Does one-millionth of a cubic inch seem very small? Each side would measure .01 of an inch. Do you think you could see it without a magnifying glass? _____

It might look like a grain of sand. But there are things much smaller. If this tiny piece one-millionth of a cubic inch were hollow like a box, it could hold many bacteria — about 15 million of them. Molecules are much smaller than bacteria. Atoms are still smaller. Electrons are still smaller —

In arithmetic, "of" and "times" indicate the same idea. And we have several ways to write fractions. Complete the following:

$\frac{1}{10}$ of $\frac{1}{1} = \frac{1}{10}$	.1 of 1 = ____
$\frac{1}{10}$ of $\frac{1}{10} = \frac{1}{\quad}$	.1 of .1 = ____
$\frac{1}{10}$ of $\frac{1}{100} =$ ____	.1 x .01 = ____
$\frac{1}{100} \times \frac{1}{100} =$ ____	.01 x .01 = ____
$\frac{1}{10} \times \frac{1}{1{,}000} =$ ____	.1 x .001 = ____
$\frac{2}{10} \times \frac{2}{10} =$ ____	.2 x .2 = ____
$\frac{2}{10} \times \frac{6}{10} =$ ____	.2 x ____
$\frac{5}{100} \times \frac{5}{100} =$ ____	.05 x ____
$\frac{3}{10} \times \frac{15}{1{,}000} =$ ____	.3 x ____

Other names for 1:

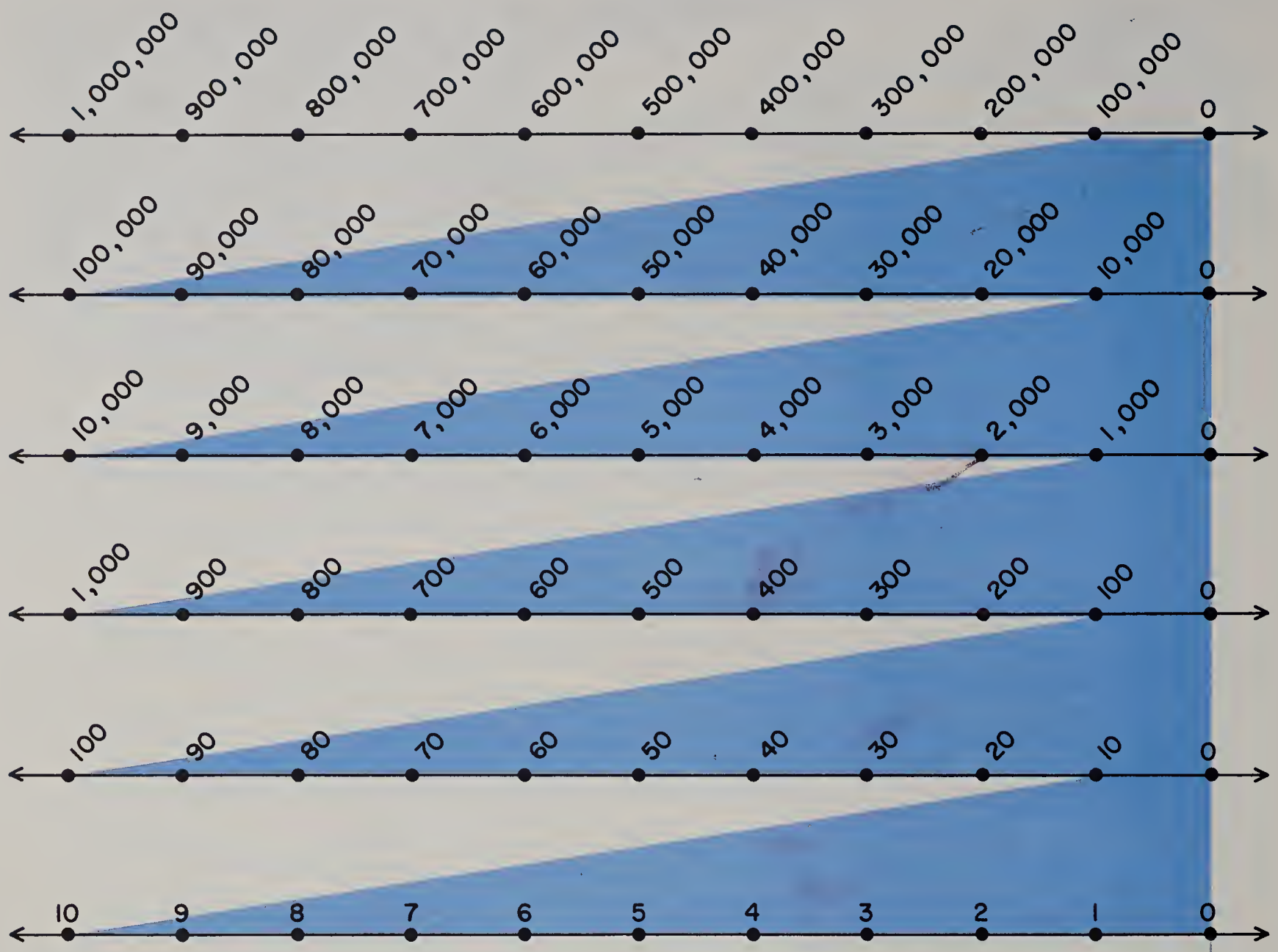
.3 + ____	.1 x ____	.7 ÷ ____
1.4 − ____	____ ÷ .19	____ + .08
____ + 1.0	____ − .012	.25 x ____
____ x 5	.07 + ____	10 x ____
8 − ____	____ x .01	2.3 − ____
100 x ____	____ + .19	____ ÷ 93
____ − 1.81	____ x .5	____ + .84
____ x .025	.007 + ____	.02 x ____

Other names for .1:

.2 ÷ ____	____ + .06	____ − 1.3
1 x ____	.2 x ____	____ ÷ 10
____ x .001	____ x 10	.011 + ____

Other names for .01:

.01 + ____	____ x 1	.8 − ____
.001 + ____	10 x ____	5 x ____
____ − .123	.009 + ____	____ x .0025
2 x ____	____ + .0001	100 x ____



It's easy to read large numbers:

79 , 347 , 205 , 870

Billions Millions Thousands

Start reading the numbers at the left, and use the indicated words as you come to each comma:

"Seventy nine billion, three hundred forty-seven million, two hundred five thousand, eight hundred seventy."

. . . unless there are 3 zeros to the left of a comma. Then, don't say anything when you pass the 3 zeros and the comma that follows it.

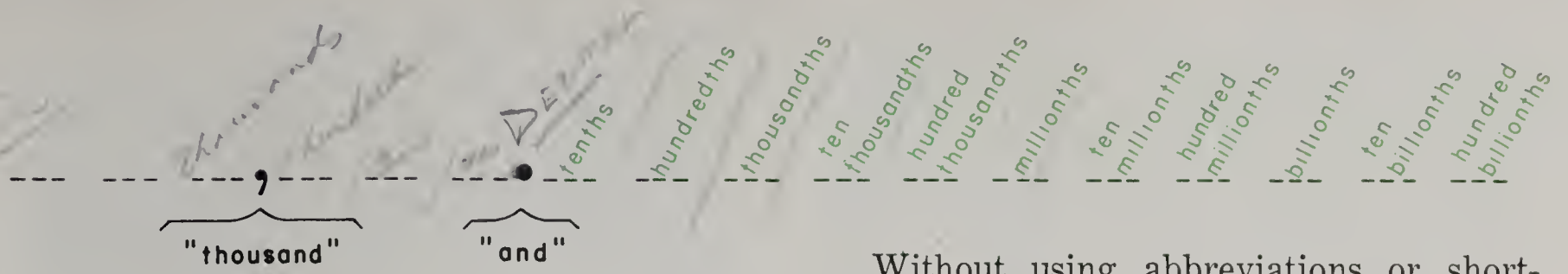
2 , 120 , 000 , 061

Billions Millions Pass over

"Two billion, one hundred twenty million, sixty-one."

Read the following numbers out loud and locate them on a number line — unless they are starred (*).

3	3,087
18	750,762
905	8,400,036 *
1,000	1,001,358,265 *
1,001	71,000
80,000	8,000,762 *
48,206	127,138,000 *
175,175	1,000,001
301,000	100,000,984 *
800,008	60,001
51,000	100,000
7,871	876,543
999,999	1,000,100
1,000,000	1,700,000 *



The "decimal point" is another bit of shorthand. It is read "and" if the number is more than one.

3.7 . . . is read, "three and seven tenths".

108.18 . . . is read, "one hundred eight and eighteen hundredths".

8,176.04 . . . is read, "eight thousand one hundred seventy-six and four hundredths".

When there are no significant figures to the left of the decimal point, we do not pronounce the "and".

.9 . . . is read, "nine tenths"

0.12 . . . is read, "twelve hundredths"

.084 . . . is read, "eighty - four thousandths"

.1072 . . . is read, "one thousand seventy-two ten thousandths"

Caution: confusion can result if "and" is used for any other reason than to indicate the location of the decimal point. A careless person might read the last example above as

"one thousand and seventy-two ten thousandths" . . . which means

1,000.0072

Read the following expressions:

a	3.08	h	.1078
b	18.1	i	.010
c	4.125	j	2.100
d	.008	k	4.001
e	.1009	l	.0807
f	129.018	m	.19473
g	27.00012	n	.002009

Without using abbreviations or shorthand, write out the expressions given at the bottom of the last column.

a _____

_____

b _____

_____

c _____

_____

d _____

e _____

_____

f _____

_____

g _____

_____

h _____

_____

i _____

j _____

_____

k _____

_____

l _____

_____

m _____

_____

_____

n _____

_____

When Mr. Dean walked into the arithmetic class one day, he saw the chart below on the board — the part shown in black. Jenny was calling out numbers.

"Four," she said, and wrote 4 at the top of the third column from the left.

The others wrote numbers opposite their names in the same column.

"I've got the rules," Henry spoke up. "After Mary's name and under 10, I would write 17. In the same column, I would write 20 after Bill's name, 9 after Alice's name, 21 after George's name, and 30 after Al's name."

Numbers Jennie called out	3	1	4	10	9	Rules:	Shorthand *n*
Mary	10	8	11			Add seven to the number called out	*n*+7
Bill	6	2	8			Double the number called out	2*n*
Alice	2	0	3			Subtract one from the number called out	*n*−1
George	7	3	9			Double the number called out and add one to the result	2*n*+1
Al	9	3	12			Multiply the number called out by three	3*n*

(Complete the record in the fourth and fifth columns.)

Because Henry had discovered the rules in the game above, he took Jennie's place and called out the numbers. He selected ten members of the class to make up secret rules. Can you discover the rules from the records given? Can you find a shorthand expression for the rules?

Numbers called out	2	9	25	7	13	Rules:	Shorthand *a*
Joe	4	18	50				2*a*
Bob	0	7	23				*a*−2
Catherine	5	19	51	15			
Sandra	13	20	36				
Sue	3	17	49	13			
Bill	6	27					
Dorothy	7	21	53				
June	9	37	101				
Ed	4	25	73				
Helen	11	46	126				

A Pattern Hunt

I

3	→	8
0	→	5
10	→	15
5	→	10
7	→	12
9	→	

$n \rightarrow n+5$

20	→	25
81	→	
7.5	→	
$\frac{1}{2}$	→	
	→	

II

7	→	14
23	→	46
0	→	0
1	→	
2	→	
8	→	

$a \rightarrow 2a$

19	→	38
27	→	
4.3	→	
$\frac{1}{3}$	→	
	→	

III

0	→	0
1	→	3
2	→	6
10	→	30
7	→	
19	→	

$s \rightarrow$

28	→	
50	→	
	→	
	→	
	→	

IV

19	→	17
86	→	84
2	→	0
7	→	5
21	→	
50	→	

$r \rightarrow r-$

	→	
	→	
	→	
	→	
	→	

V

2	→	5
5	→	11
7	→	15
0	→	1
6	→	
4	→	

$g \rightarrow 2g+$

10	→	
50	→	
$\frac{1}{3}$	→	
	→	
	→	

VI

0	→	0
3	→	21
10	→	70
5	→	35
9	→	
13	→	

$h \rightarrow$

28	→	
	→	308
	→	91
	→	$3\frac{1}{2}$
	→	

VII

1	→	12
19	→	30
11	→	22
7	→	18
8	→	
26	→	

$t \rightarrow$

49	→	
	→	$12\frac{1}{2}$
	→	100
	→	
	→	

VIII

3	→	16
7	→	24
0	→	10
25	→	60
9	→	
17	→	

$x \rightarrow 2x+$

$\frac{1}{2}$	→	
	→	15
	→	$10\frac{1}{2}$
	→	
	→	

IX

20	→	0
5	→	15
17	→	3
10	→	10
12	→	
19	→	

$y \rightarrow 20-$

0	→	
	→	19
	→	$7\frac{1}{2}$
	→	$5\frac{2}{3}$
	→	

X

10	→	107
0	→	7
5	→	57
$\frac{1}{2}$	→	12
20	→	207
2	→	27

$z \rightarrow$

8	→	
	→	257
	→	97
.2	→	
	→	

XI

5	→	25
7	→	49
3	→	
2	→	
8	→	
11	→	

$n \rightarrow n \cdot n$

10	→	100
	→	
	→	
	→	
	→	

XII

5	→	20
7	→	42
3	→	6
1	→	0
10	→	
9	→	

$b \rightarrow b \cdot b-$

11	→	
20	→	
	→	210
	→	72
	→	

XIII

14	→	7
9	→	$4\frac{1}{2}$
50	→	25
4	→	
28	→	
31	→	

→

58	→	
174	→	
	→	95
	→	$\frac{1}{2}$
	→	

XIV

1	→	.7
4	→	2.8
.1	→	.07
20	→	14
10	→	
100	→	

→

	→	
	→	
	→	
	→	
	→	

XV

3	→	5
10	→	19
7	→	13
50	→	99
5	→	
9	→	

→

59	→	
	→	
	→	
	→	
	→	

1.

$n \rightarrow 3n + 3$

5 → 18	1 → ___
0 → 3	10 → ___
2 → ___	25 → ___
6 → ___	13 → ___

2.

$x \rightarrow 3(x + 1)$

1 → ___	0 → ___
10 → ___	2 → ___
25 → ___	6 → ___
13 → ___	5 → ___

3.

$y \rightarrow y \cdot y - 1$

5 → ___	___ → ___
9 → ___	___ → ___
6 → ___	___ → ___
1 → ___	___ → ___

4.

$s \rightarrow 10s + 5$

11 → ___	___ → ___
8 → ___	___ → ___
0 → ___	___ → ___
5 → ___	___ → ___

5.

$t \rightarrow (2t + 1)5$

11 → ___	___ → ___
8 → ___	___ → ___
0 → ___	___ → ___
5 → ___	___ → ___

6.

$b \rightarrow (b + 1)(b - 1)$

5 → ___	___ → ___
9 → ___	___ → ___
6 → ___	___ → ___
1 → ___	___ → ___

7.

$a \rightarrow a(a + 1)$

0 → ___	___ → ___
1 → ___	___ → ___
4 → ___	___ → ___
9 → ___	___ → ___

8.

$c \rightarrow c^2 + c$

0 → ___	___ → ___
1 → ___	___ → ___
4 → ___	___ → ___
9 → ___	___ → ___

9.

$m \rightarrow 7m + 13$

2 → ___	___ → ___
5 → ___	___ → ___
8 → ___	___ → ___
11 → ___	___ → ___

10.

$n \rightarrow 32 - 4n$

1 → ___	$\frac{1}{2}$ → ___
3 → ___	___ → ___
5 → ___	___ → ___
0 → ___	___ → ___

11.

$q \rightarrow 4(8 - q)$

1 → ___	___ → ___
3 → ___	___ → ___
5 → ___	___ → ___
0 → ___	___ → ___

12.

$r \rightarrow 13 + 7r$

2 → ___	___ → ___
5 → ___	___ → ___
8 → ___	___ → ___
11 → ___	___ → ___

13.

$d \rightarrow 10 + 8d - 2$

3 → ___	___ → ___
8 → ___	___ → ___
10 → ___	___ → ___
5 → ___	___ → ___

14.

$k \rightarrow 8(k + 1)$

3 → ___	___ → ___
8 → ___	___ → ___
10 → ___	___ → ___
5 → ___	___ → ___

15.

$f \rightarrow 2(5 + 4f - 1)$

3 → ___	___ → ___
8 → ___	___ → ___
10 → ___	___ → ___
5 → ___	___ → ___

A Pattern Hunt

A

2	2	4
0	0	0
3	6	18
4	1	4
10	5	50
0	9	
6		24

a	*b*	*ab*
	9	54

B

2	2	4
0	0	0
3	6	9
4	1	5
	9	28
	11	77
40		81

x	*y*	*x+*
14	41	
27	72	
	51	100
$\frac{1}{4}$	$\frac{1}{2}$	

C

4	2	2
0	0	0
12	2	10
17	9	8
28		18
	5	29
	81	12

r	*s*	*r-*
$\frac{1}{2}$	$\frac{1}{4}$	
81	18	
	100	12

D

2	4	2
5	10	2
3	18	6
5	50	10
2	1	$\frac{1}{2}$
4	2	
50	5	

p	*q*	
9	27	
	34	17

E

10	15	5
7	11	4
29	30	1
4	18	14
27	50	
	50	27

m	*n*	
16	93	
	68	41

F

3	0	6
3	1	7
5	0	10
5	3	13
0	4	4
1	10	12

c	*d*	
4	7	15
6	4	
3	9	
11	5	

G

5	10	17
9	3	14
50	25	77
0	0	2
0	3	5
10	0	12

x	*a*	
4	9	
	6	31
	18	97

H

4	4	1
10	5	6
25	10	16
7	0	8
100	75	26
18	9	10

m	*y*	
27	4	
	6	19
	11	51

I

2	3	16
1	9	19
5	5	35
0	0	10
1	1	11
7	7	59

c	*d*	
9	7	
	6	46
		41
		69

J

10	0	5
18	0	9
10	1	6
18	1	10
14	1	8
22	11	22

x	*y*	
100	7	57
	7	20
	17	87

K

3	7	20
9	3	24
25	25	100
7	23	60
$\frac{1}{2}$	$\frac{1}{2}$	2
8	7	

a	*b*	
	7	34
	$7\frac{1}{2}$	48
	8	66

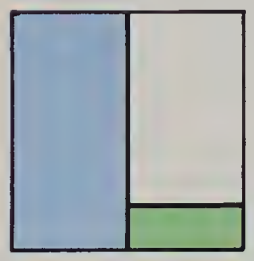
L

3	0	9
2	0	4
4	0	16
3	1	10
3	2	11
4	100	116

s	*t*	
7	1	50
8	5	
3	11	

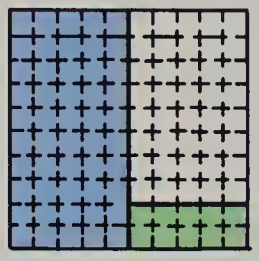
Let us agree that each of the diagrams below suggests the idea of 1. Then the shaded and unshaded areas suggest fractions of 1 — fractions that are less than 1. Consider each, and complete the record at the bottom of the page.

a.

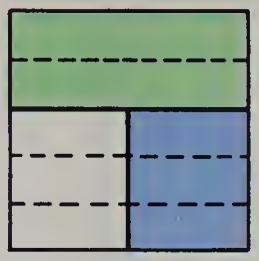


b.

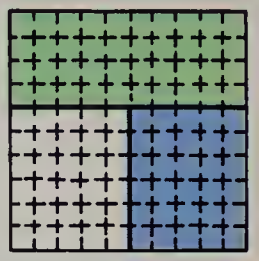
c.



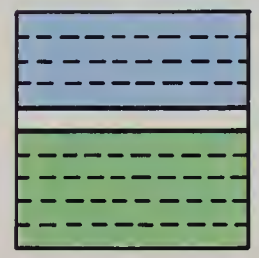
d.



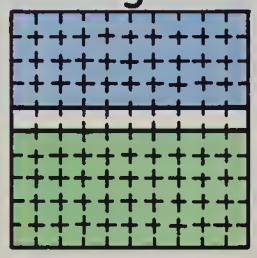
e.



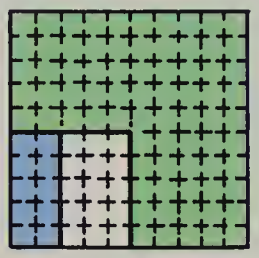
f.



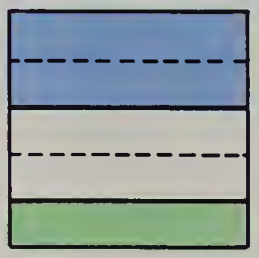
g.



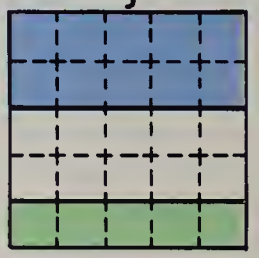
h.



i.



j.



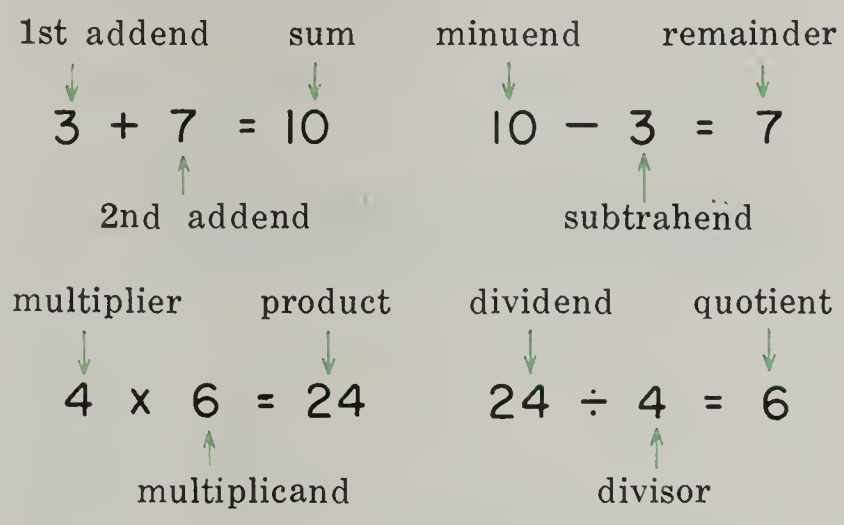
	Part shaded Blue		Part shaded Green		Unshaded Part		Blue and Green together		Green and unshaded together	
	common fraction	decimal fraction	common fraction	decimal fraction	common fraction	decimal fraction	common fraction	decimal fraction	common fraction	decimal fraction
a.										
b.										
c.										
d.										
e.										
f.										
g.										
h.										
i.										
j.										

In arithmetic, we are usually adding or subtracting, multiplying or dividing. We are often faced with the job of completing such mathematical sentences as:

$9 \times 29 = ___$ or $81 + ___ = 367$

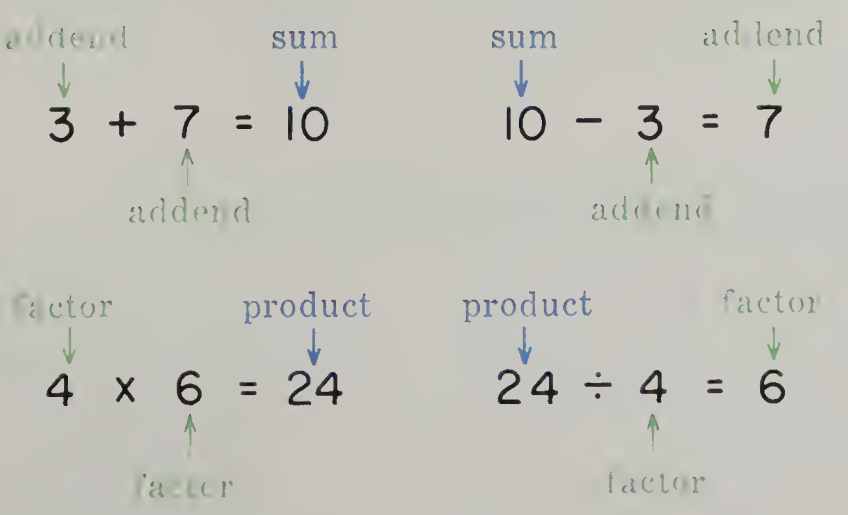
or $___ - 2\frac{1}{2} = 7$ or $10 \div 3 = ___$

Sometimes we want to talk about certain parts of those sentences. In the past, a very clumsy language was used. It is suggested in the following examples:



A much simpler and more useful language has been recommended by the School Mathematics Study Group. The recommended language has a further advantage in that it makes clear that addition and subtraction are simply slightly different ways of talking about the same idea, as are — for a different idea — multiplication and division.

The recommended language uses only four words instead of the 12 above.



"Addend," "sum," "factor," "product" . . . that's all.

In addition and subtraction, we are concerned with two addends and a sum. Often one of them is missing. What is the missing term, and what is its name?

We are looking for the missing—

$17 + 96 = ___$	sum
$18 + ___ = 50$	addend
$___ + 19 = 91$	
$10 - 3 = ___$	
$37 - ___ = 9$	
$___ - 12 = 41$	

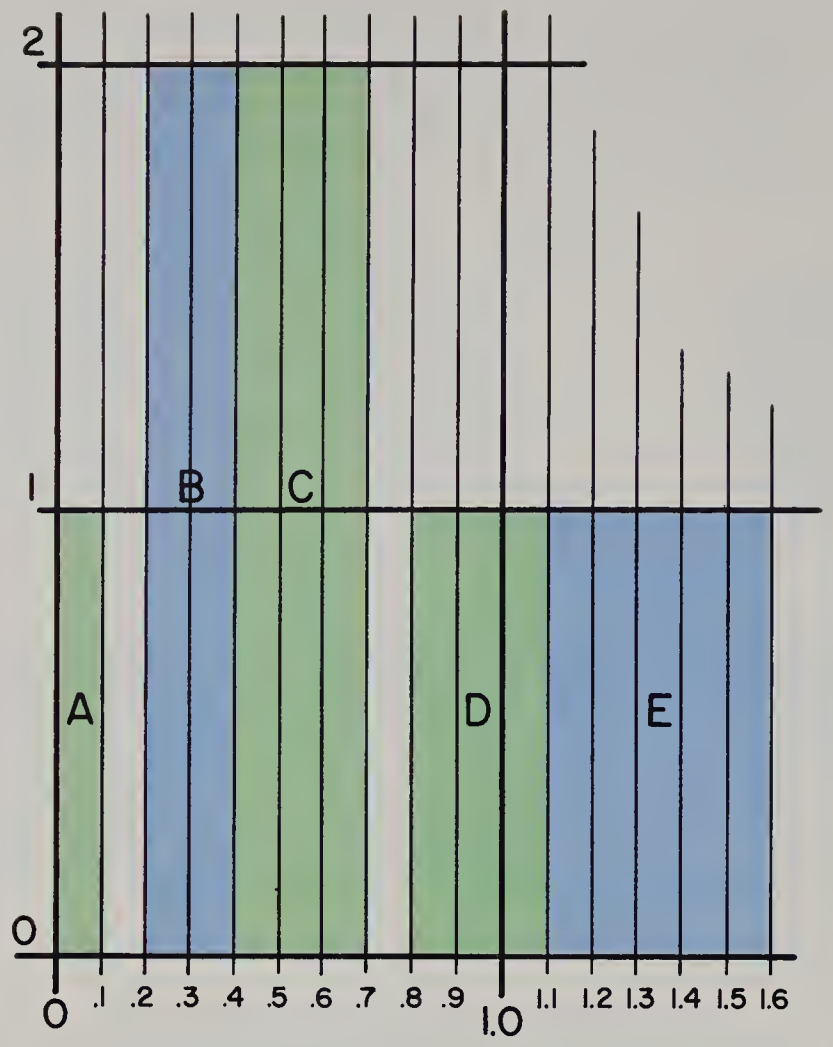
In multiplication and division, we are concerned with two factors and a product.

We are looking for the missing—

$9 \times 5 = ___$	product
$___ \div 5 = 9$	
$___ \times 3\frac{1}{2} = 7$	
$100 \div ___ = 25$	
$10 \div 3 = ___$	
$8 \times ___ = 400$	

Use "a," "s," "f," or "p" to indicate missing term.
(Also complete each as a true statement)

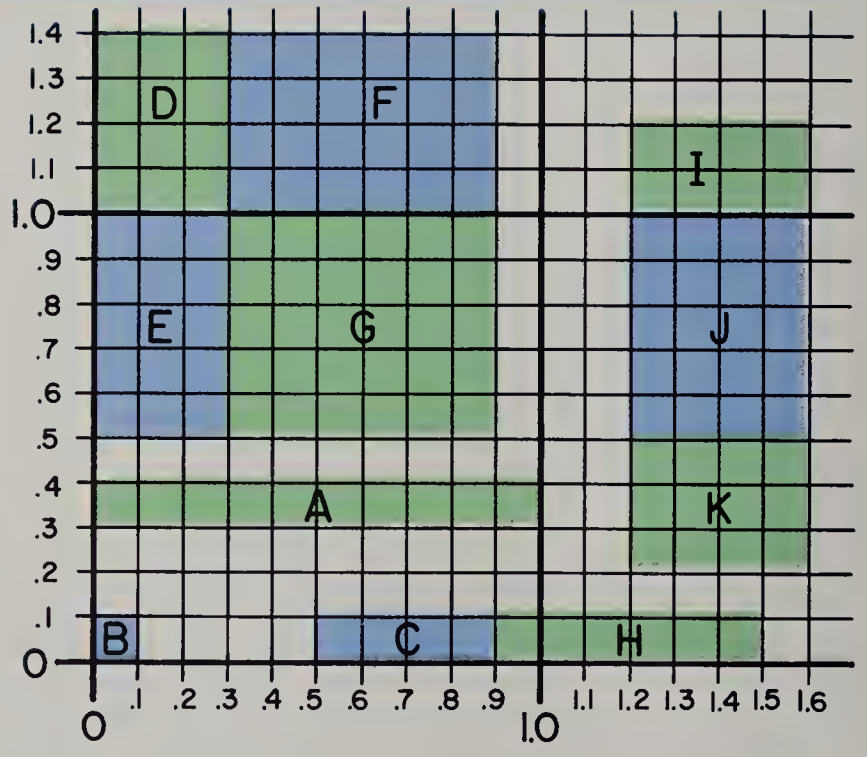
$13 \times 11 = ___$	p	$___ \times \frac{1}{2} = 81$	
$\frac{1}{2} + \frac{1}{2} = ___$		$___ - \frac{1}{3} = \frac{2}{3}$	
$300 \div 25 = ___$		$___ \div 17 = 7$	
$118 - ___ = 77$		$100 \times 100 = ___$	
$___ \div 19 = 8$		$375 + ___ = 1{,}000$	
$___ - 425 = 375$		$1{,}000 \div ___ = 8$	
$4 \times ___ = 144$		$87 = 100 - ___$	



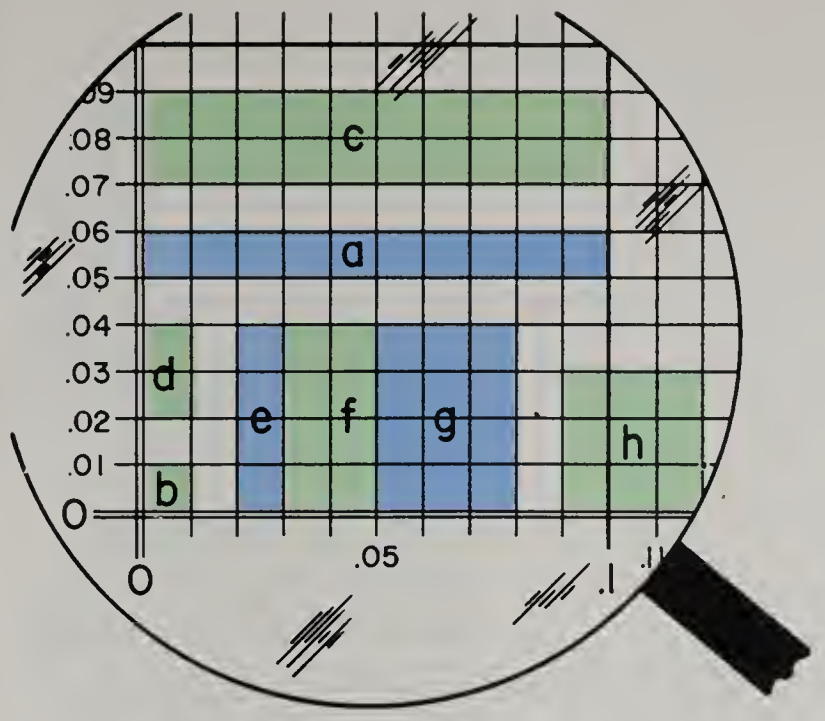
Diagrams Can Suggest Important Ideas

(Note: In this section, B + C means the picture of area B and area C placed side by side. C − B means the difference in the areas of the two.)

Part of the diagram	Some of the ideas suggested by the part or parts of the diagram indicated
A	1 x .1 , .1 x ___ , .1
B	2 x ___ , ___ x 2 , .4
C	___ x ___ , ___ x ___ , ___
D	___ x ___ , ___ x ___ , ___
B + C	2 (.2 + ___) , .5 x ___ , ___ + ___ , ___ , ___
D + E	1 (___ + ___) ,
E − D	



Part of the diagram	Some of the ideas suggested by parts of the diagram
A	.1 x ___ , ___ x ___ , ___
B	.1 x .1 , ___
C	
D	
E	
C + H	
F + G	.4 x .6 + .5 x .6 , (___ + ___) .6 .30 + ___ , ___
I + J + K	.4 (___ + ___ + ___) , .___ .08 + ___ + ___ ,
D + I + F	.4 (___ + ___ + ___) , ___ + ___ + ___ ,
D + E + F + G	(.4 + ___) (.3 + ___) , ___ + ___ + ___ + ___ , .3 (.4 + ___) + .6 (.4 + ___)



Under a Magnifying Glass—

This sketch indicates how the little blue block B on the previous page might look under a strong magnifying glass. It is a sketch to suggest that .1 × .1 = .01.

On each number line shown, .1 is divided into 10 parts. Each then is $\frac{1}{10}$ of $\frac{1}{10}$ or $\frac{1}{100}$ or .01.

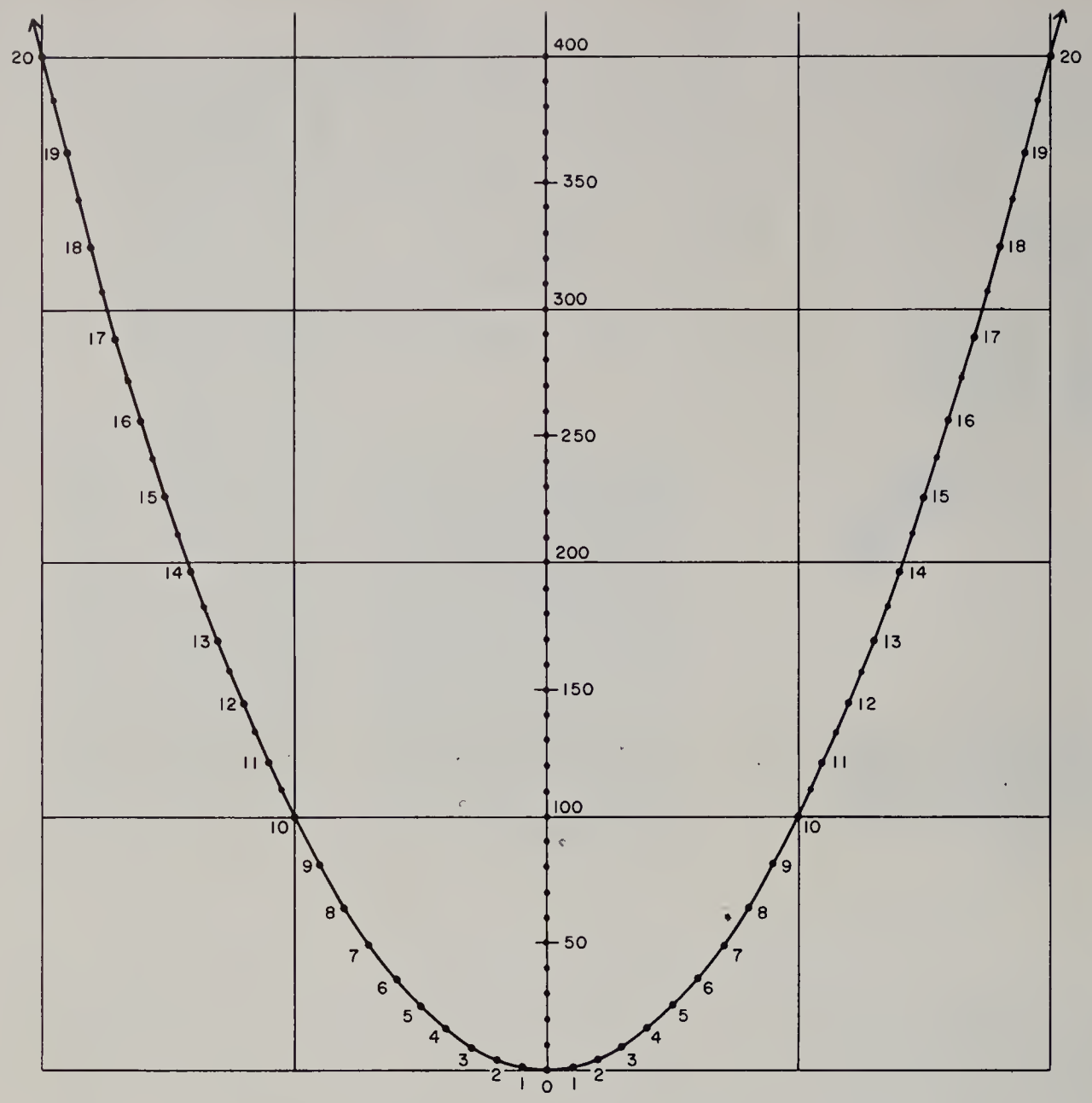
Parts of the diagram	Some of the ideas suggested by parts indicated
a	.01 x .1 , .1 x .01 , .001
b	.01 x .01 , .____
c	.02 x .1 , .1 x ___ , .____
d	.02 x .01 , .01 x ___ , .____
e	
f	
g	
h	
b + h	
b + d	
a + c	
a + e	
f + g	.04(___ + ___) , _____ _____ + _____
e + f + g	(.01 + .02 + ___)___ , _____ _____ + _____ + _____

Complete this little Adding Machine

+	3	.4	2.5	.03
4	7	4.4	6.5	
.3	3.3			
.9	3.9			
3.7				
.16				
1.6				
.54				
2.5				
.07				
.67				
2.79				
1.97				

A Multiplying Machine

×	4	.4	.04	1.5	3.25	2.01	.35
2							
.6							
.2							
.01							
.12							



When two factors are given, find one on each of the lines that curve up and out from 0. Connect these points with a straight-edge. At what point does it seem to cut the center scale?

Where one factor is missing, lay a straight-edge across the graph through the point indicating the given factor and the product. Notice where the straight-edge cuts the other curved line.

If you work carefully, you can estimate the size of the missing factor or product within a small amount.

In the statement, $12 \times 15 = 180$, we call 12 and 15 the "factors," and we call 180 the "product" of those factors. Is it true that, if any two of the three elements of such a statement are given, we can find the third?

one factor	15	9		17	18
another factor	10		16	17	
product		108	192		270

The incompleted statement, $162 \div 18 = ___$ indicates that the product and one of the factors is given. To complete it, the missing factor must be found.

Find the missing factor in each:

$12 \times ___ = 144$ $\quad$ $306 \div 18 = ___$

$13 = 182 \div ___$ $\quad$ $___ \times 8 = 104$

Missing Terms

	using the graph	by computation
$12 \times 16 = ___$		
$___ \times 17 = 187$		
$13 \times ___ = 195$		
$247 \div 19 = ___$		
$324 \div ___ = 18$		
$___ \div 19 = 12$		
$10.5 \times 18 = ___$		
$12 \times 17.5 = ___$		
$216 \div 16 = ___$		
$___ \div 8 = 19.5$		
$10.5 \times 10.5 = ___$		
$361 \div 19 = ___$		
$13 \times 130 = ___$		
$1.5 \times 17 = ___$		
$1.3 \times 1.6 = ___$		
$297 \div 16.5 = ___$		
$158 \div ___ = 12\frac{2}{13}$		

This is the same diagram that appeared on the previous page, but the scale has been changed. If both factors are less than 100, you can use it to estimate your answers.

The dash line illustrates a division example:

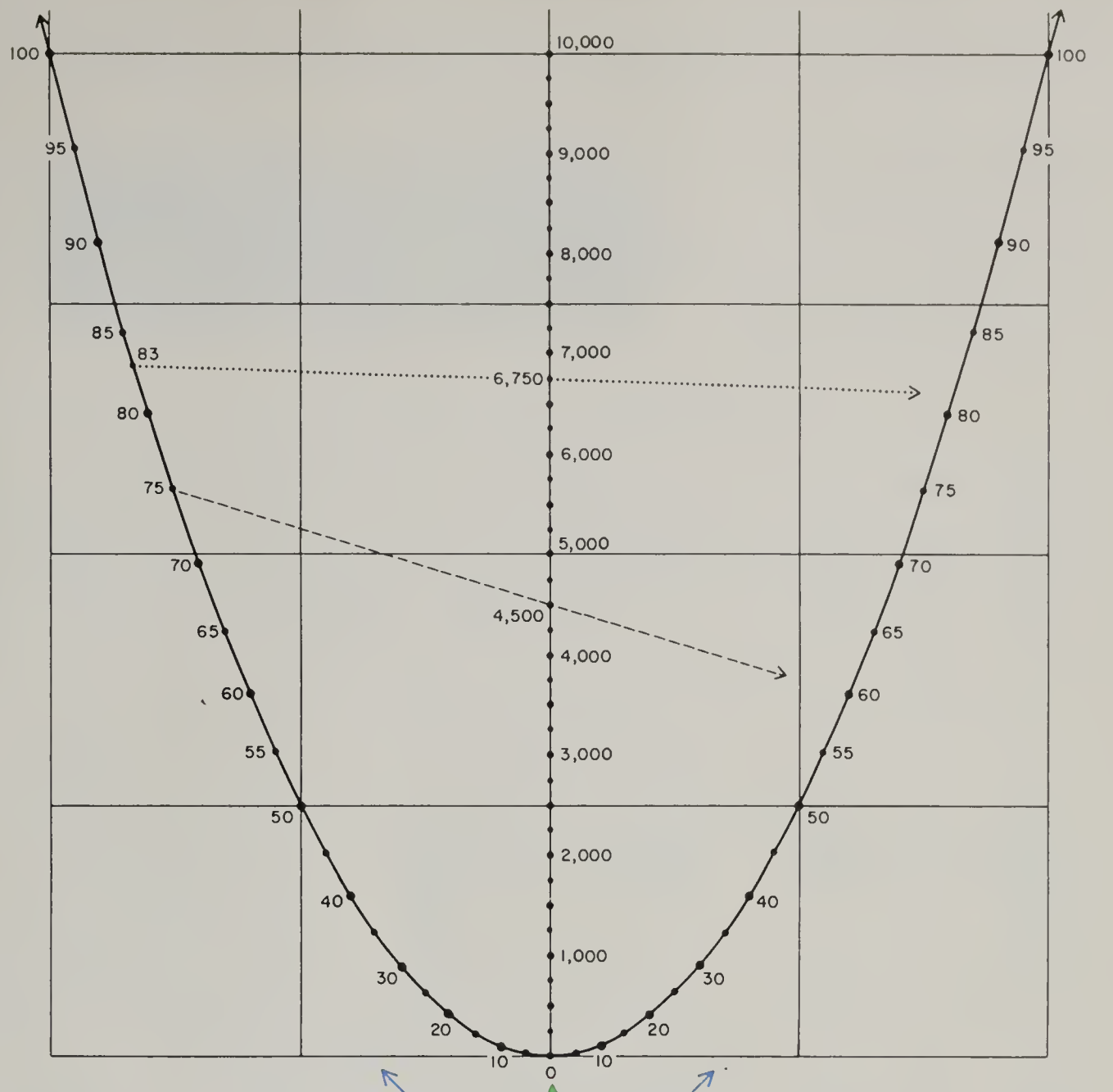
4,500 ÷ 75 = ____

You know that the answer is about 60. If you work out the arithmetic you will find how close your estimate was. (Exact, in this case.) But try another.

The dotted line above the dash line illustrates another division example:

6,750 ÷ 83 = ____

You know the answer is about ____.



	A	B	C	D	E	F	G	H
product	4,500	2,400	8,100	3,750	3,600	2,025	3,575	5,625
factor	75	80	90	75	45	45	65	75
factor	60							

	estimate	result of computation
I		
J		
K		
L		
M		
N		
O		
P		
Q		

Estimate the answer to each example below and record it on the left in the first column. Then find the exact answer and compare them.

I. 35)840 J. 68)4,216 K. 59)3,009

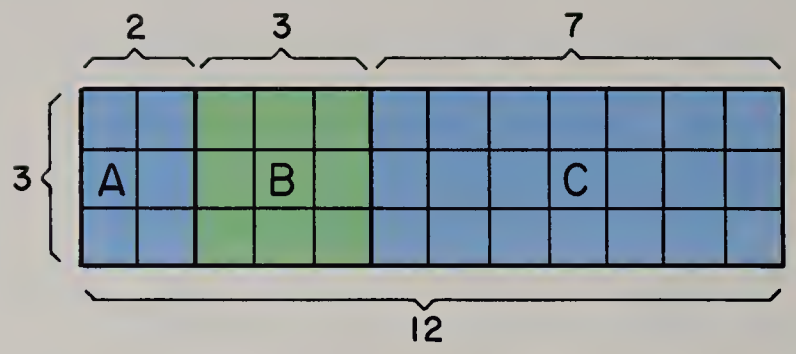
L. 51)1,530 M. 85)7,225 N. 74)5,476

O. 52)468 P. 73)6,716 Q. 67)4,221

Copy the graph above. See if you can work out other scales that would be helpful.

Diagrams often suggest useful mathematical ideas.

A. 3 · 2 = ___
B. 3 · 3 = ___
C. 3 · 7 = ___
3 · 12 = ___



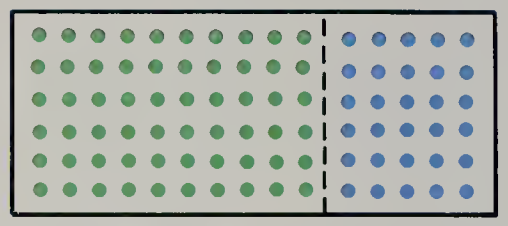
A. 6 ÷ 3 = ___
B. 9 ÷ 3 = ___
C. 21 ÷ 3 = ___
36 ÷ 3 = ___

---

6 · 10 = ___
6 · ___ = ___
6 · ___ = 90

or

___ · 6 = ___
___ · 6 = ___
___ · 6 = 90



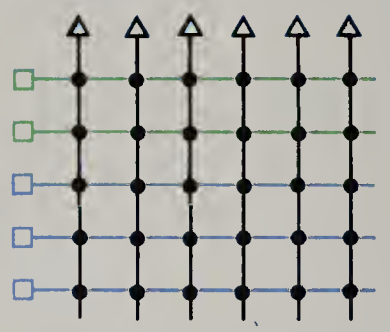
60 ÷ 6 = ___
30 ÷ 6 = ___
90 ÷ 6 = ___

---

2 · ___ = ___
3 · ___ = ___
5 · ___ = ___

or

6 · ___ = ___
6 · ___ = ___
6 · ___ = ___



12 ÷ 6 = ___
18 ÷ 6 = ___
___ ÷ 6 = 5

You draw the sketches

4 · 3 = ___
4 · 6 = ___
4 · 9 = ___

___ ÷ 4 = ___
___ ÷ 4 = ___
___ ÷ 4 = ___

---

5 · 3 = ___
8 · 3 = ___
___ · 3 = 39

___ ÷ 3 = ___
___ ÷ 3 = ___
___ ÷ 3 = ___

---

6 · 5 = ___
2 · 5 = ___
4 · 5 = ___
___ · 5 = ___

___ ÷ 5 = ___
___ ÷ 5 = ___
___ ÷ 5 = ___
___ ÷ 5 = ___

---

2 · 10 = ___
2 · 3 = ___
2 · ___ = ___

___ ÷ 2 = ___
___ ÷ 2 = ___
___ ÷ 2 = ___

Examples without pictures

a.

___ · 3 = 15	___ ÷ 3 = ___
___ · 3 = 21	___ ÷ 3 = ___
___ · 3 = 36	36 ÷ 3 = ___

b.

___ · 6 = 18	___ ÷ 6 = ___
___ · 6 = ___	___ ÷ 6 = ___
___ · 6 = 48	48 ÷ 6 = ___

c.

___ · 5 = 20	20 ÷ 5 = ___
___ · 5 = 35	35 ÷ 5 = ___
___ · 5 = ___	___ ÷ 5 = ___
___ · 5 = 100	100 ÷ 5 = ___

d.

___ · 7 = ___	___ ÷ 7 = 20
___ · 7 = ___	___ ÷ 7 = ___
___ · 7 = ___	___ ÷ 7 = 9
___ · 7 = 273	273 ÷ 7 = ___

e.

17 · 10 = ___	___ ÷ ___ = 10
17 · ___ = ___	___ ÷ ___ = ___
17 · ___ = 238	238 ÷ 17 = ___

f.

___ · 23 = ___	___ ÷ 23 = 30
___ · 23 = ___	___ ÷ 23 = ___
___ · 23 = 713	713 ÷ 23 = ___

A Game: In the following examples, only multiples of 10 are to be written in the spaces tinted green; only numbers less than 10 in the spaces tinted blue. For example:

g.

20 · 5 = 100	100 ÷ 5 = 20
7 · 5 = 35	35 ÷ 5 = 7
27 · 5 = 135	135 ÷ 5 = 27

h.

___ · 8 = 80	___ ÷ 8 = ___
___ · 8 = 24	___ ÷ 8 = ___
___ · 8 = 104	___ ÷ 8 = ___

i.

6 · ___ = ___	___ ÷ 6 = 20
6 · ___ = ___	___ ÷ 6 = 8
6 · 28 = ___	___ ÷ 6 = 28

j.

7 · ___ = ___	___ ÷ 7 = ___
7 · ___ = ___	___ ÷ 7 = ___
7 · ___ = 77	77 ÷ 7 = ___

k.

___ · 9 = ___	___ ÷ 9 = ___
___ · 9 = ___	___ ÷ 9 = ___
___ · 9 = 207	207 ÷ 9 = ___

l.

___ · 11 = ___	___ ÷ 11 = ___
___ · 11 = ___	___ ÷ 11 = ___
___ · 11 = 363	363 ÷ 11 = ___

m.

___ · 25 = ___	___ ÷ ___ = ___
___ · 25 = ___	___ ÷ ___ = ___
___ · 25 = 625	625 ÷ 25 = ___

n.

___ · ___ = ___	___ ÷ ___ = ___
___ · ___ = ___	___ ÷ ___ = ___
13 · ___ = 559	559 ÷ 13 = ___

o.

___ · ___ = ___	___ ÷ ___ = ___
___ · ___ = ___	___ ÷ ___ = ___
___ · ___ = 657	657 ÷ ___ = ___

(Use 1 as a factor only when you can find no alternative.)

On this page, observe the rules of the game outlined on the previous page and, in the green spaces with a black line across them, write a fraction that is less than 1 and has no common factor in the numerator and denominator.

■ · 12 = 120	120 ÷ ___ = ■
■ · 12 = 60	60 ÷ ___ = ■
$\frac{1}{3}$ · 12 = 4	4 ÷ 12 = —
___ · 12 = 184	184 ÷ 12 = $15\frac{1}{3}$

A

■ · 8 = ■	80 ÷ 8 = ■
■ · 8 = ___	64 ÷ 8 = ■
— · 8 = 4	___ ÷ 8 = —
___ · 8 = 148	148 ÷ 8 = ___

B

■ · ___ = ■	■ ÷ 14 = ■
■ · ___ = ___	___ ÷ 14 = ■
— · ___ = ___	___ ÷ 14 = —
___ · 14 = 352	352 ÷ 14 = ___

In the examples below, only one part of each pattern is given.

C

■ ÷ 16 = ■	200 ÷ 16 = ___
___ ÷ 16 = ■	
___ ÷ 16 = —	

D

■ · 21 = ■	___ · 21 = 364
■ · 21 = ___	
— · 21 = ___	

E

■ · 15 = ■	___ · 15 = 228
■ · 15 = ___	
— · 15 = ___	

F

■ ÷ 4 = ■	258 ÷ 4 = ___
___ ÷ 4 = ■	
___ ÷ 4 = —	

G

■ ÷ ___ = ■	635 ÷ 25 = ___
___ ÷ ___ = ■	
___ ÷ ___ = —	

H

■ ÷ ___ = ■	448 ÷ 12 = ___
___ ÷ ___ = ■	
___ ÷ ___ = —	

I

■ ÷ ___ = ■	411 ÷ 18 = ___
___ ÷ ___ = ■	
___ ÷ ___ = —	

J

■ ÷ 9 = ■	183 ÷ 9 = ___
___ ÷ 9 = ■	
___ ÷ 9 = —	

K

■ ÷ ___ = ■	459 ÷ 36 = ___
___ ÷ ___ = ■	
___ ÷ ___ = —	

L

■ ÷ ___ = ■	287 ÷ 28 = ___
___ ÷ ___ = ■	
___ ÷ ___ = —	

M

■ ÷ ___ = ■	435 ÷ 45 = ___
___ ÷ ___ = ■	
___ ÷ ___ = —	

N

■ ÷ 37 = ■	481 ÷ 37 = ___
___ ÷ 37 = ■	
___ ÷ 37 = —	

O

■ ÷ ■ = 42	700 ÷ ___ = 42
___ ÷ ■ = 42	
___ ÷ — = 42	

P

■ ÷ ■ = 8	900 ÷ ___ = 8
___ ÷ ■ = 8	
___ ÷ — = 8	

Q

■ ÷ ■ = 77	1,023 ÷ ___ = 77
___ ÷ ■ = 77	
___ ÷ — = 77	

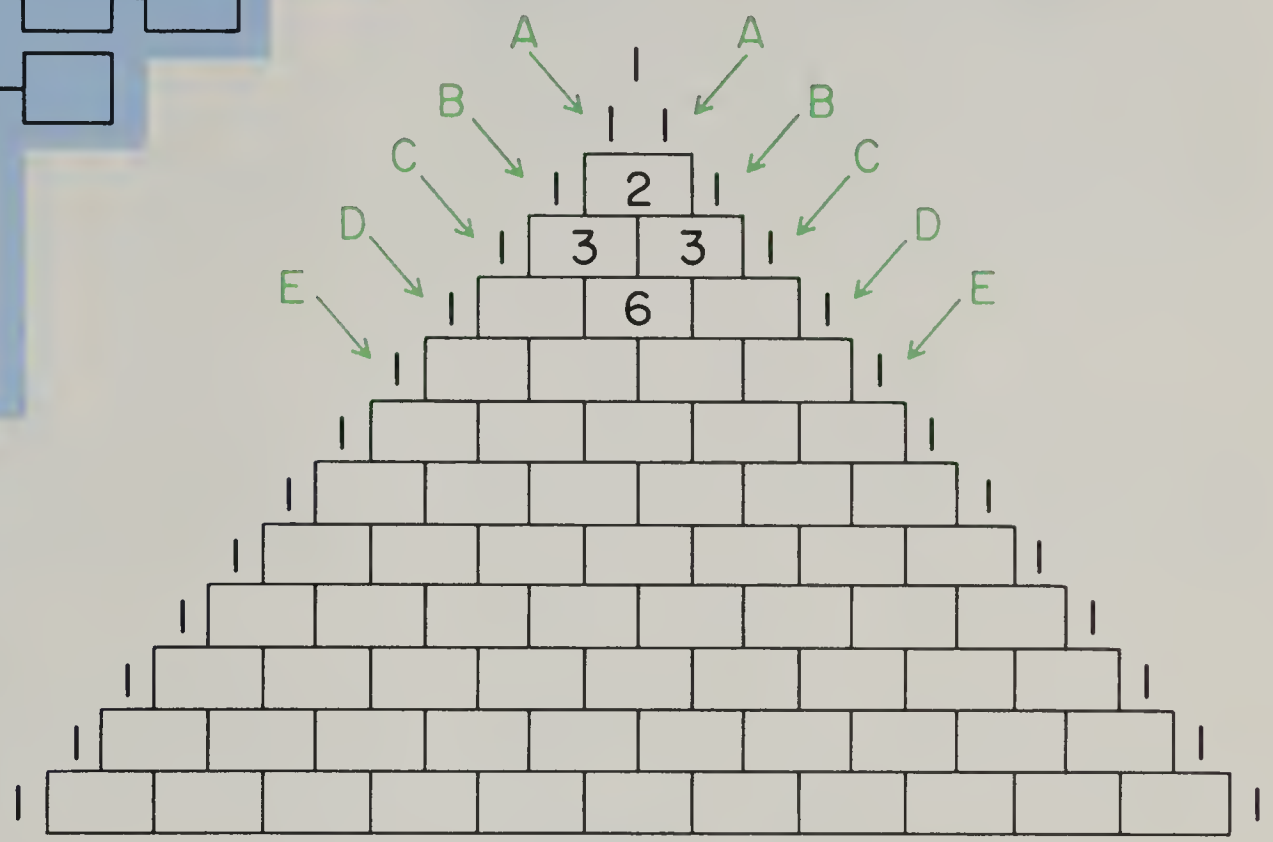
R

■ ÷ ■ = 27	855 ÷ ___ = 27
___ ÷ ■ = 27	
___ ÷ — = 27	

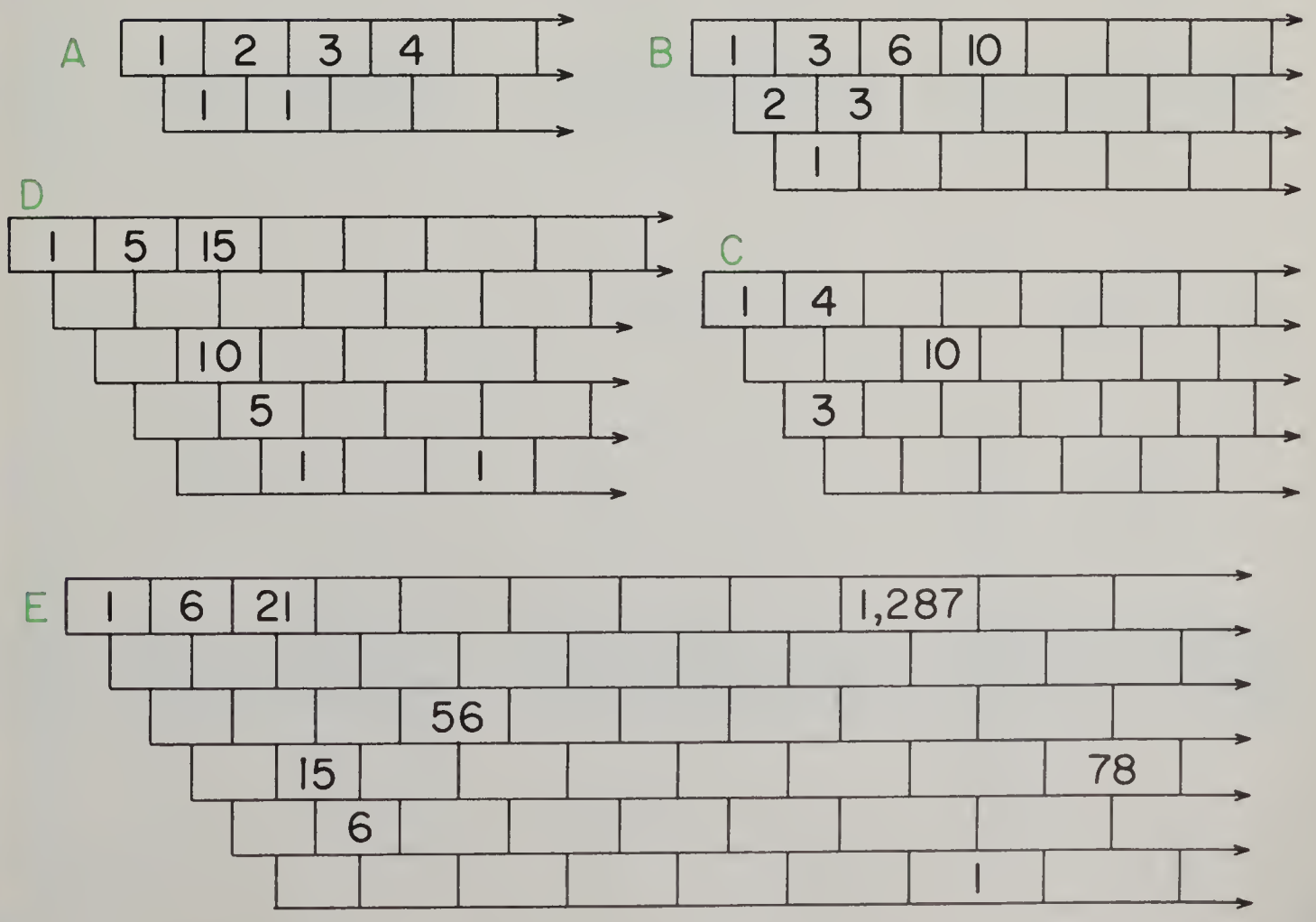
Each number written after column A is one less than twice the number written on its left. Column A contains all even numbers, 50 and less.

A	B	C	D	E	F	G
2	3	5	9			
4	7	13				
6	11					
8						
10						
12						
14						
16						

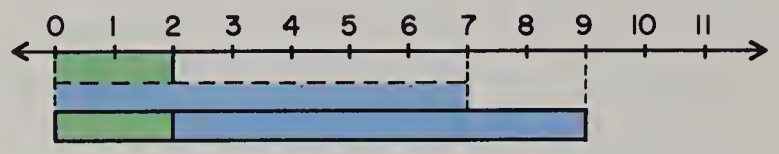
In the triangular-shaped sketch below, the numbers indicated in each box are the sum of the two numbers above it — one on the right and one on the left.



Except for the top row in the examples below, each number written in a box is the difference of the number written above it on the right and the left. Each top row is a series indicated by an arrow above.



Here is one of the many pictures that suggests the idea of addition and subtraction.



This picture suggests the idea just as well as those below. Each expression is simply another way of saying the same thing: we are arranging the addends and sum in a different order.

$7 + 2 = 9$ $\qquad$ $9 - 7 = 2$

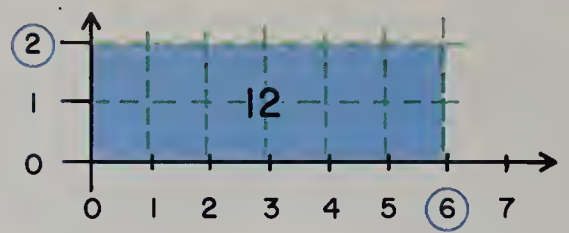
$2 + 7 = 9$ $\qquad$ $9 - 2 = 7$

And, by switching the "addition" and "equal" signs, we change the form but not the meaning:

$9 = 7 + 2$ $\qquad$ $2 = 9 - 7$

$9 =$ _____ $\qquad$ $7 =$ _____

Here is one of the many pictures that suggests the idea of multiplication and division.



The factors are 2 and 6; the product is 12. (If you are in doubt, count the small blocks in the blue-tinted area.) This picture illustrates all of the following:

___ x ___ = ___ $\qquad$ ___ ÷ ___ = ___

___ x ___ = ___ $\qquad$ ___ ÷ ___ = ___

And, by switching the "multiplication" and "equal" signs, we change the form but not the meaning:

___ = ___ x ___ $\qquad$ ___ = ___ ÷ ___

___ = ___ x ___ $\qquad$ ___ = ___ ÷ ___

Pattern Hunt

a

7 , 0 → 7
19 , 91 → 110
6 , 8 → 14
65 , 65 → ____
42 , 72 → ____

a , b → a + b

$\frac{1}{2}$, $\frac{1}{4}$ → ____
.2 , .3 → ____

missing sum

b

2 , 2 → 4
1 , 1 → 1
0 , 0 → 0
8 , 3 → 24
9 , 12 → ____

m , n → m · n

____ , ____ → ____
____ , ____ → ____

missing product

c

4 , 2 → 2
9 , 3 → 3
70 , 10 → 7
42 , 6 → ____
96 , 12 → ____

x , y → x ÷

____ , ____ → ____
____ , ____ → ____

missing _____

d

8 , 0 → 8
4 , 2 → 2
19 , 4 → 15
74 , 47 → ____
92 , 38 → ____

r , s → r −

____ , ____ → ____
____ , ____ → ____

missing _____

e

5 , 11 → 6
9 , 21 → 12
31 , 50 → 19
0 , 19 → ____

a , x → x −

____ , ____ → ____
____ , ____ → ____

missing _____

f

3 , 12 → 4
5 , 45 → 9
1 , 71 → 71
14 , 42 → ____

m , r →

____ , ____ → ____
____ , ____ → ____

missing _____

g

17 , 3 → 51
5 , 11 → 55
12 , 12 → ____
10 , 10 → ____

d , t →

20 , 20 → ____
30 , 40 → ____

missing _____

h

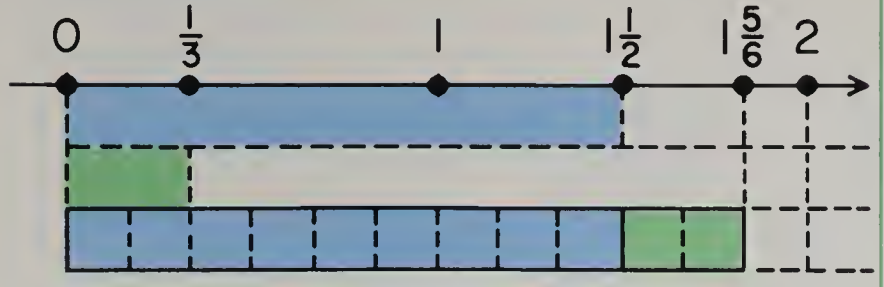
2 , 4 → 2
0 , 0 → 0
1 , 1 → 0
13 , 13 → ____

x , y →

17 , 63 → ____
$\frac{1}{3}$, 1 → ____

missing _____

A picture that suggests an example of the idea of addition and subtraction of fractions is:

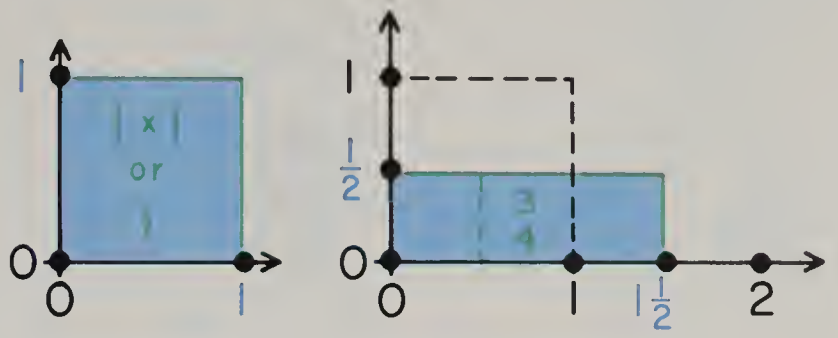


This picture suggests an idea that can be expressed in many ways. The form of the sentence changes, but not its meaning:

$1\frac{1}{2} + \frac{1}{3} = 1\frac{5}{6}$ (or $\frac{11}{6}$) $\quad 1\frac{5}{6} - \frac{1}{3} =$ ___

$\frac{1}{3} + 1\frac{1}{2} =$ ___ $\quad 1\frac{5}{6} -$ ___ $=$ ___

A picture that suggests an example of the idea of multiplication and division of fractions is:



If the sketch on the left above suggests that $1 \times 1 = 1$, then the other sketch suggests that $\frac{1}{2} \times 1\frac{1}{2} = \frac{3}{4}$. Changing the sentence form involving these factors and products, we say:

$\frac{1}{2}$ x $1\frac{1}{2} = \frac{3}{4}$ $\quad \frac{3}{4} \div \frac{1}{2} =$ ___

$1\frac{1}{2}$ x ___ $=$ ___ $\quad$ ___ $\div$ ___ $=$ ___

Addends are ___ and ___. Sum is ___. Various ways of expressing the idea suggested are:

$\frac{1}{6} + \frac{1}{3} =$ ___ $\quad$ ___ $-$ ___ $=$ ___

___ $+$ ___ $=$ ___ $\quad$ ___ $-$ ___ $=$ ___

Factors are ___ and ___. Product is ___.

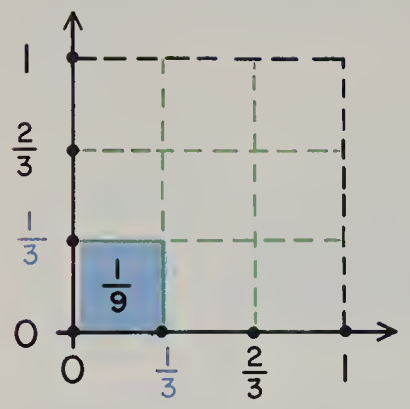
$\frac{1}{3}$ x $\frac{1}{2} =$ ___ $\quad$ ___ $\div$ ___ $=$ ___

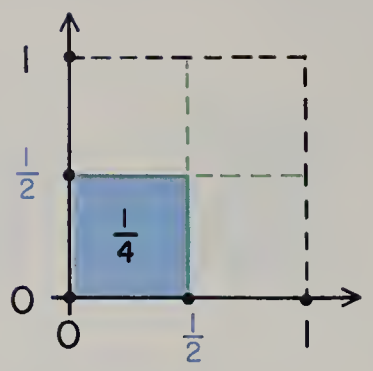
___ x ___ $=$ ___ $\quad$ ___ $\div$ ___ $=$ ___

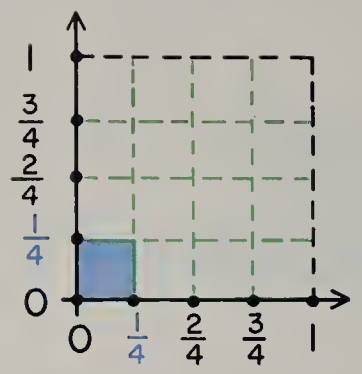
Pattern Hunt

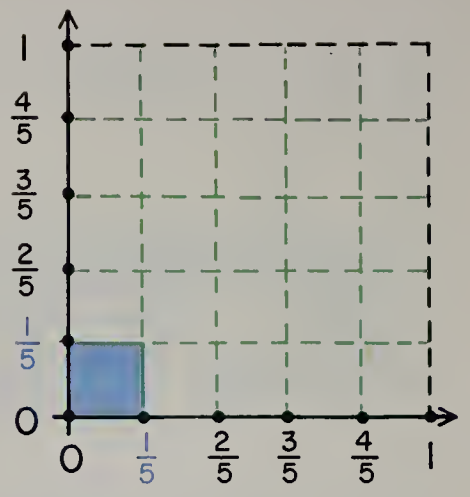
$2\frac{1}{2}, 1\frac{1}{2} \rightarrow 4$	$3, 2 \rightarrow 1\frac{1}{2}$	$\frac{1}{2}, \frac{1}{3} \rightarrow \frac{1}{6}$	$6, 8 \rightarrow 2$
$3, 8 \rightarrow 11$	$75, 15 \rightarrow 5$	$0, \frac{7}{8} \rightarrow 0$	$\frac{1}{3}, 1 \rightarrow \frac{2}{3}$
$19, 17 \rightarrow$ ___	$144, 12 \rightarrow$ ___	$7, 19 \rightarrow 133$	$1\frac{1}{2}, 8 \rightarrow 6\frac{1}{2}$
$1\frac{1}{2}, \frac{1}{3} \rightarrow$ ___	$\frac{3}{4}, \frac{1}{2} \rightarrow$ ___	$1\frac{1}{2}, \frac{1}{2} \rightarrow$ ___	$\frac{1}{3}, 1\frac{5}{6} \rightarrow$ ___
$58, 94 \rightarrow$ ___	$\frac{1}{6}, \frac{1}{3} \rightarrow$ ___	$15, 15 \rightarrow$ ___	$\frac{1}{3}, \frac{1}{2} \rightarrow$ ___
$a, n \rightarrow a + n$	$x, y \rightarrow x \div y$	$v, t \rightarrow v \cdot t$	$c, d \rightarrow d - c$
$\frac{1}{3}, 1\frac{1}{2} \rightarrow$ ___	$\frac{3}{4}, 1\frac{1}{2} \rightarrow$ ___	$\frac{1}{3}, \frac{1}{2} \rightarrow$ ___	$255, 500 \rightarrow$ ___
$\frac{1}{3}, \frac{1}{6} \rightarrow$ ___	$132, 11 \rightarrow$ ___	$13, 13 \rightarrow$ ___	$.9, 1 \rightarrow$ ___

The Sketches Suggest:









$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$\frac{1}{9} \div \frac{1}{3} = \frac{1}{3}$

$\times \quad = \quad$

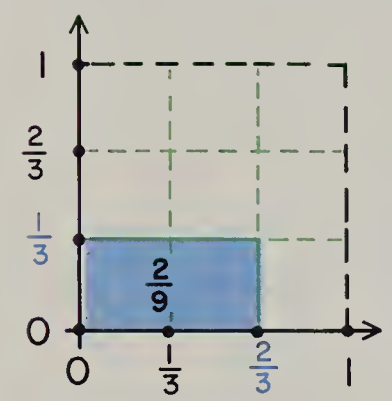
$\div \quad = \quad$

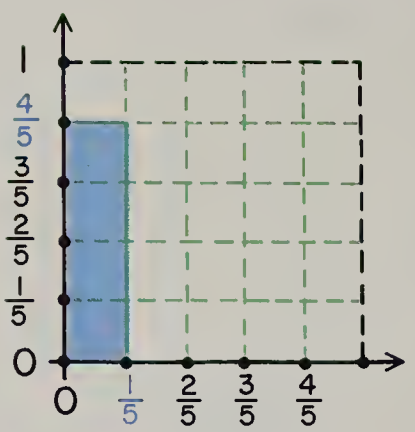
$\times \quad = \quad$

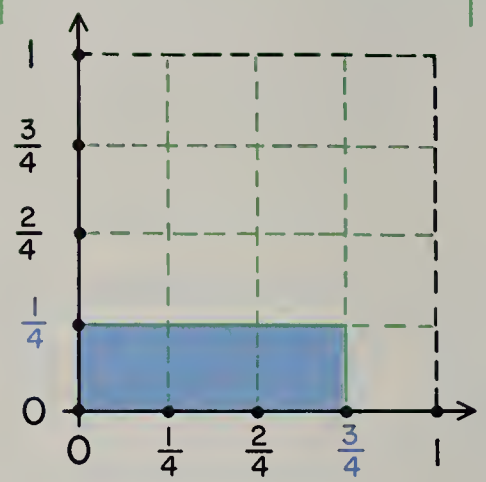
$\div \quad = \quad$

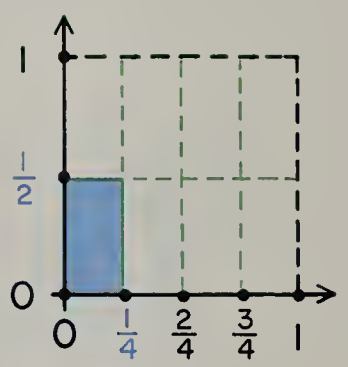
$\times \quad = \quad$

$\div \quad = \quad$









$\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

$\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

$\div \quad = \quad$

$\div \quad = \quad$

$\times \quad = \quad$

$\times \quad = \quad$

$\div \quad = \quad$

$\div \quad = \quad$

$\times \quad = \quad$

$\times \quad = \quad$

$\div \quad = \quad$

$\div \quad = \quad$

$\frac{1}{4}, \frac{1}{4} \rightarrow \frac{1}{16}$

$\frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{4}$

$\frac{1}{6}, \frac{1}{6} \rightarrow \quad$

$\frac{1}{n}, \frac{1}{n} \rightarrow \frac{1}{n \cdot n}$

$\frac{1}{7}, \frac{1}{7} \rightarrow \quad$

$\frac{1}{3}, \frac{2}{3} \rightarrow \frac{2}{9}$

$\frac{1}{5}, \frac{4}{5} \rightarrow \quad$

$\frac{1}{8}, \frac{5}{8} \rightarrow \quad$

$\frac{a}{b}, \frac{c}{b} \rightarrow \frac{a \cdot c}{b \cdot b}$

$\frac{2}{9}, \frac{4}{9} \rightarrow \quad$

$\frac{1}{16}, \frac{1}{4} \rightarrow \frac{1}{4}$

$\frac{1}{25}, \frac{1}{5} \rightarrow \frac{1}{5}$

$\frac{1}{9}, \frac{1}{3} \rightarrow \quad$

$\frac{1}{x \cdot x}, \frac{1}{x} \rightarrow \frac{1}{x}$

$\frac{1}{49}, \frac{1}{7} \rightarrow \quad$

$\frac{3}{4}, \frac{1}{4} \rightarrow \frac{3}{16}$

$\frac{1}{5}, \frac{4}{5} \rightarrow \frac{4}{25}$

$\frac{2}{3}, \frac{1}{3} \rightarrow \quad$

$\frac{r}{s}, \frac{t}{s} \rightarrow \frac{r \cdot t}{\cdot}$

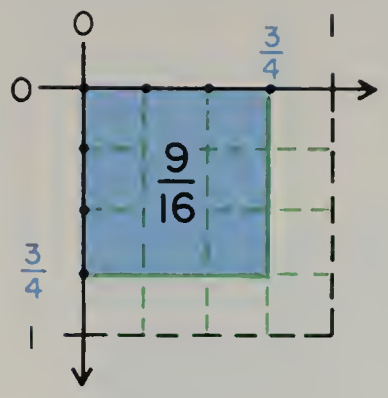
$\frac{3}{4}, \frac{3}{4} \rightarrow \quad$

missing product

missing

missing

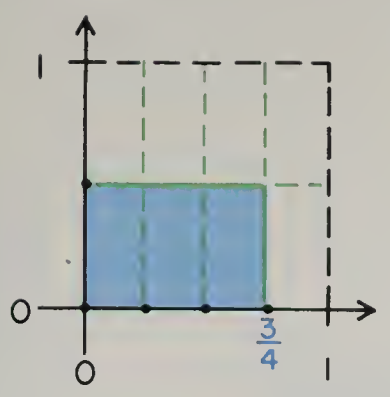
missing



$\frac{3}{4} \times \frac{3}{4} = __$

$\frac{9}{16} \div __ = __$

(Why do we show only two forms?)

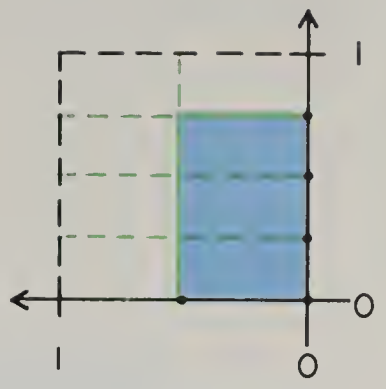


$__ \times \frac{3}{4} = __$

$__ \times __ = __$

$__ \div __ = __$

$__ \div __ = __$

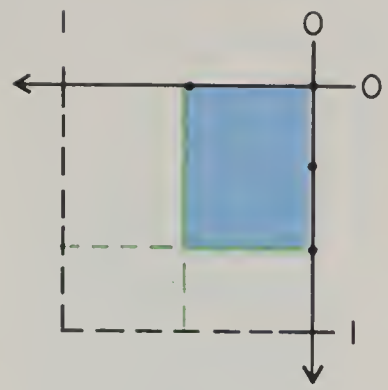


$__ \times __ = __$

$__ \div __ = __$

$__ \times __ = __$

$__ \div __ = __$



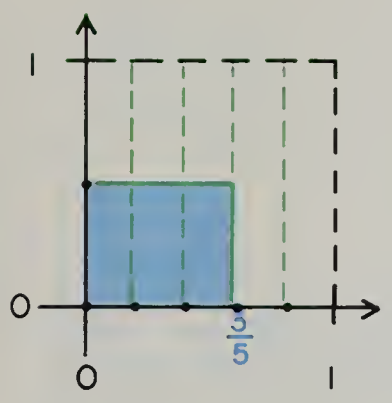
$__ \times __ = __$

$__ \div __ = __$

$__ \div __ = __$

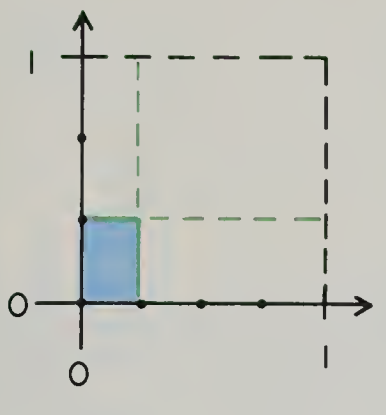
$__ \times __ = __$

Indicate only two forms of the idea suggested in each of the four examples:



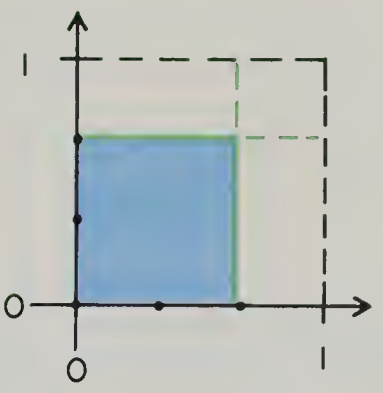
$__ \times __ = __$

$__ \div __ = __$



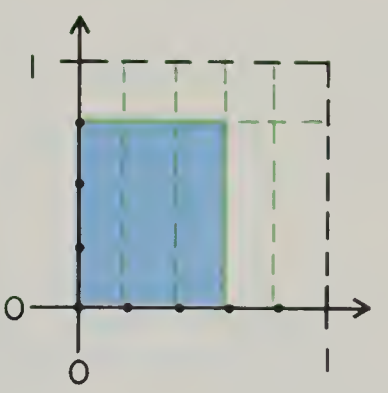
$__ \times __ = __$

$__ \div __ = __$



$__ \times __ = __$

$__ \div __ = __$



$__ \times __ = __$

$__ \div __ = __$

One way of describing patterns:

$\frac{3}{4} \times \frac{3}{4} = __$

$\frac{1}{2} \times \frac{1}{3} = __$

$\frac{3}{4} \times \frac{1}{2} = __$

$\frac{1}{3} \times \frac{1}{4} = __$

$\frac{a}{b} \times \frac{c}{d} = \frac{a \cdot c}{__}$

$\frac{2}{3} \times \frac{2}{3} = __$

$\frac{3}{4} \times \frac{3}{5} = __$

$\frac{5}{7} \times \frac{2}{3} = __$

$\frac{2}{9} \times \frac{4}{5} = __$

$\frac{9}{16} \div \frac{3}{4} = __$

$\frac{4}{9} \div \frac{2}{3} = __$

$\frac{3}{8} \div \frac{1}{2} = __$

$\frac{3}{10} \div __ = \frac{3}{5}$

$\frac{a \cdot c}{b \cdot d} \div \frac{c}{d} = __$

$__ \div \frac{2}{3} = \frac{4}{5}$

$\frac{1}{4} \div \frac{1}{2} = __$

$\frac{1}{8} \div \frac{1}{4} = __$

$\frac{1}{16} \div \frac{1}{4} = __$

$\frac{3}{5} \times \frac{3}{4} = __$

$\frac{1}{3} \times __ = \frac{1}{12}$

$__ \times \frac{2}{3} = \frac{4}{9}$

$\frac{1}{2} \times __ = \frac{3}{10}$

$\frac{x}{y} \times \frac{m}{t} = __$

$\frac{2}{5} \times \frac{4}{7} = __$

$\frac{3}{8} \times __ = \frac{3}{16}$

$\frac{3}{10} \times \frac{7}{10} = __$

$\frac{3}{4} \times __ = \frac{15}{28}$

$\frac{1}{9} \div \frac{1}{3} = __$

$\frac{9}{20} \div \frac{3}{5} = __$

$\frac{3}{10} \div __ = \frac{1}{2}$

$\frac{4}{25} \div \frac{1}{5} = __$

$\frac{x \cdot m}{y \cdot t} \div \frac{x}{y} = __$

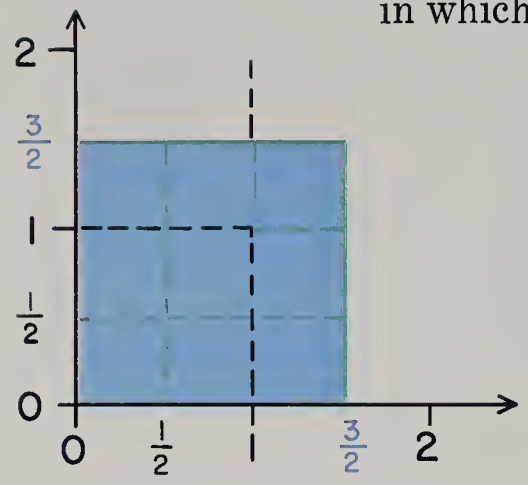
$\frac{3}{16} \div \frac{1}{4} = __$

$\frac{1}{36} \div __ = \frac{1}{6}$

$\frac{8}{15} \div __ = \frac{2}{3}$

$__ \div \frac{2}{5} = \frac{7}{11}$

These pictures suggest multiplication and division in which at least one of the factors is greater than 1.

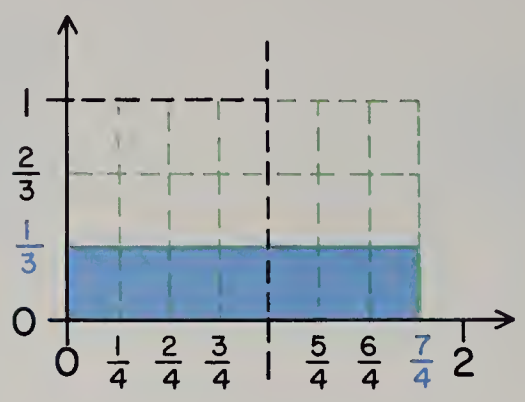


$\frac{3}{2} \times \frac{3}{2} = \quad = \quad$

$\div \quad = \quad =$

$\frac{11}{17} \times \frac{5}{3} = \quad =$

$\frac{13}{5} \times \frac{9}{7} = \quad =$

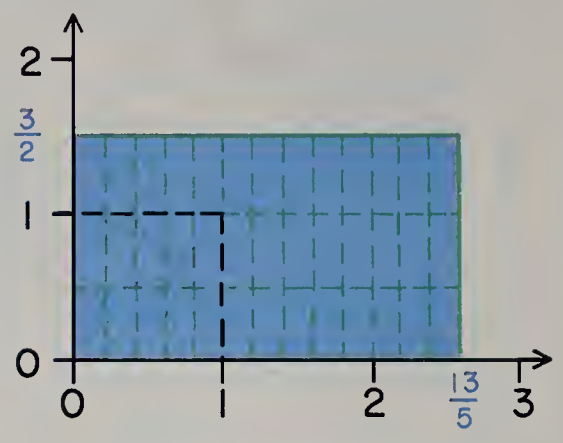


$\times \quad =$

$\times \quad =$

$\div \frac{7}{4} =$

$\div \frac{1}{3} = \quad =$



$\times \quad = \quad =$

$\times \quad = \quad =$

$\div \quad = \quad =$

$\div \quad = \quad =$

Draw as many sketches as you need to find the product of the factors (blue tint) taken in pairs, one from each blue-tinted block.

Some shorthand:

$a \times c = a \cdot c = ac$

$1 \times x = 1 \cdot x = x$

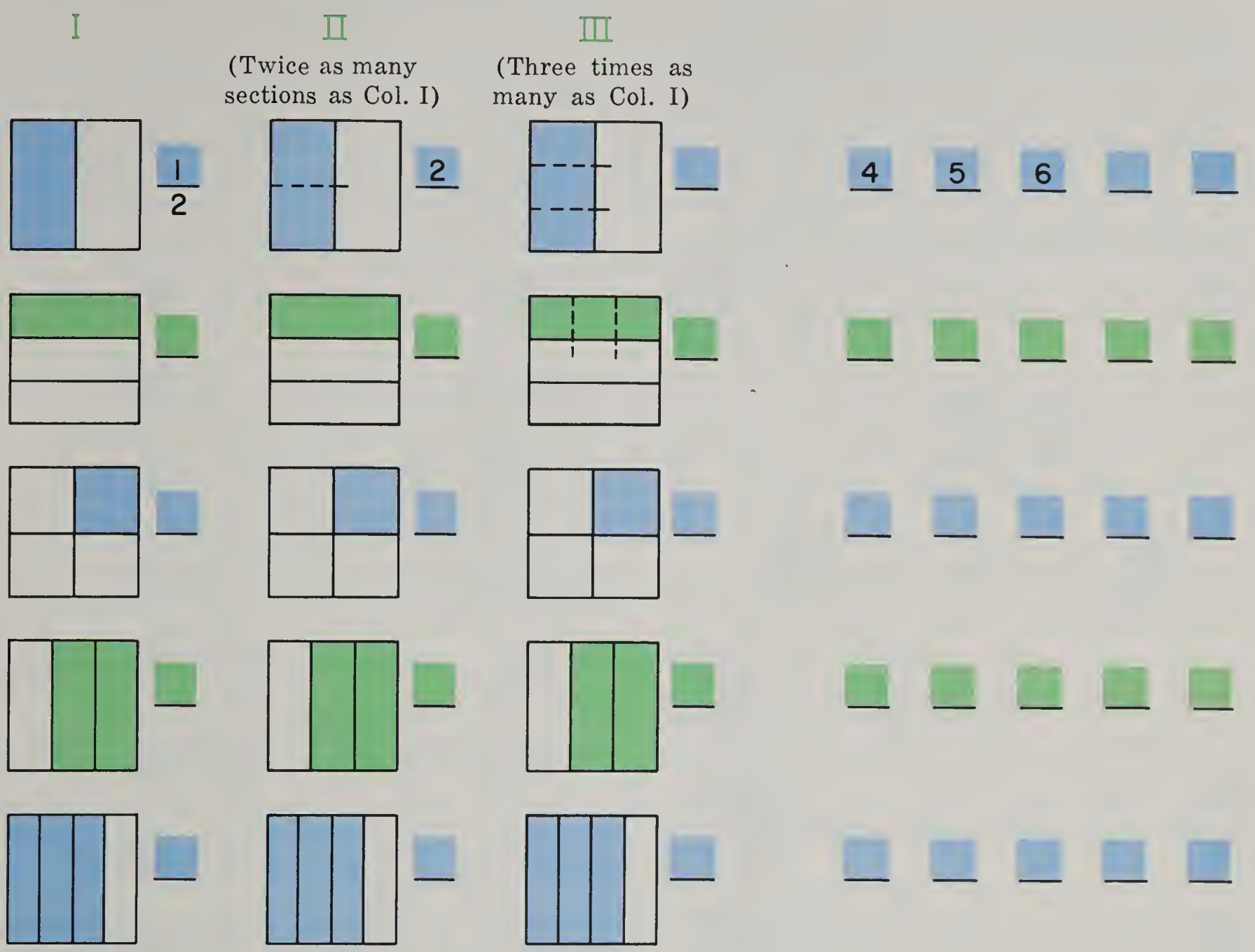
$3 \times r = 3 \cdot r = 3r$

X	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{1}{8}$	$\frac{4}{7}$	$\frac{8}{11}$	$\frac{7}{10}$	$\frac{c}{d}$	$\frac{4}{5}$	$\frac{11}{8}$	$\frac{x}{y}$	$\frac{7}{8}$	$\frac{m}{q}$
$\frac{1}{3}$	$\frac{1}{6}$							$\frac{c}{3d}$					
$\frac{9}{11}$													
$\frac{3}{5}$													
$\frac{a}{b}$	$\frac{a}{2b}$							$\frac{ac}{bd}$					
$\frac{r}{t}$			$\frac{2r}{5t}$										

Make up examples of your own that fit each of the patterns suggested below:

$\frac{d}{m} \times \frac{r}{h} = \frac{dr}{\quad}$	$\frac{1}{x} \times \frac{1}{y} = \frac{1}{\quad}$	$\frac{2}{3} \times \frac{s}{s} = \quad$	$\frac{4a}{1} \times \frac{1}{5a} = \quad$
$\frac{2}{3} \times \frac{4}{7} = \quad$	$\frac{1}{4} \times \frac{1}{6} = \quad$	$\frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$	$\frac{4 \cdot 3}{1} \times \frac{1}{5 \cdot 3} = \frac{12}{15}$
$\quad \times \frac{1}{3} = \frac{5}{6}$	$\frac{1}{7} \times \frac{1}{11} = \quad$	$\frac{2}{3} \times \frac{8}{8} = \quad$	$\frac{4 \cdot 6}{1} \times \frac{1}{5 \cdot 6} = \quad$
$\quad \times \frac{5}{11} = \frac{10}{88}$	$\quad \times \frac{1}{3} = \frac{1}{6}$	$\frac{2}{3} \times \frac{4}{4} = \quad$	$\frac{4 \cdot 9}{1} \times \frac{1}{5 \cdot} = \quad$
$\frac{7}{8} \times \quad = \frac{63}{80}$	$\frac{1}{5} \times \quad = \frac{1}{50}$	$\quad \times \frac{6}{6} = \quad$	$\frac{4 \cdot 5}{1} \times \quad = \quad$

In Column II, add lines so that the sketch is divided into twice as many sections as the sketch in Column I. In Column III, add lines so that the sketch is divided into three times as many sections as in Column I. In the small tinted block, indicate the number of tinted sections; and below it, the total number of sections. On the right, predict results if the activity were continued in the same pattern.



Without the aid of pictures, record your predictions of the results of the following extensions of the idea developed above:

I	II	III	4	5	6	7	...	10	11	...	24	25	...	n
1/5	2/10	3/15	4	5										
3/7	6/14	9												
4/5	8/10													
8/9	16/18													
6/11	12													

A In the previous study, we developed an idea that might be summarized as follows:

$$\frac{4}{7} = \frac{40}{70} = \frac{32}{56} = \frac{8}{\quad} = \frac{20}{\quad} = \frac{\quad}{49} = \frac{4a}{7a}$$

Of the many names for a fraction, there is one in which the numerator and denominator have no common whole-number factors except 1. That is shown above in the green tint. Also, there is a way of indicating the whole family that names the same fraction. That is indicated in the blue tint.

$$\frac{3}{5} = \frac{9}{15} = \frac{\quad}{50} = \frac{12}{\quad} = \frac{\quad}{30} = \frac{\quad}{25} = \frac{\quad}{5x}$$

$$\frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{7n}{9n} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$$

$$\frac{\quad}{\quad} = \frac{b}{\quad} = \frac{\quad}{\quad} = \frac{66}{36} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$$

$$\frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{24}{90} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{s}$$

$$\frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{65}{40} = \frac{\quad}{\quad} = \frac{\quad}{\quad} = \frac{\quad}{\quad}$$

$$\frac{\quad}{\quad} = \frac{85}{95} = \frac{\quad}{76} = \frac{136}{\quad} = \frac{\quad}{380} = \frac{\quad}{\quad}$$

C Marvin Hawkins claimed he had found a way to tell quickly whether two fractions are equal or not. All he would say was the following:

$\frac{15}{23} \neq \frac{9}{15}$ because $225 \neq 207$

$\frac{8}{13} = \frac{40}{65}$ because $520 = 520$

Then he put the following examples on the board to see if others could discover his rule. Can you?

$\frac{7}{9} \neq \frac{9}{12}$ because $\quad \neq 81$

$\frac{6}{10} = \frac{21}{35}$ because $\quad =$

$\frac{8}{14} \;\square\; \frac{36}{63}$ because $\quad \;\square\; \quad$

$\frac{14}{15} \;\square\; \frac{11}{12}$ because $\quad \;\square\; \quad$

$\frac{25}{45} = \quad$ because $1{,}350 = 1{,}350$

$\frac{24}{\quad} = \frac{33}{\quad}$ because $1{,}056 = 1{,}056$

$\frac{\quad}{21} \;\square\; \frac{\quad}{17}$ because $306 \neq 315$

$\frac{\quad}{18} = \frac{\quad}{26}$ because $702 = 702$

B Tom asked Mary, Barney and Arlene to play a game. The results of the game can be summarized as follows:

Tom said:	Mary thought:	Barney thought:	Arlene thought:	Tom thought:
"Think of a fraction."	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{6}{11}$	$\frac{a}{b}$
"Think of any whole number."	9	3	4	r
"Multiply the numerator and denominator of the fraction by the number you thought of."	$\frac{3 \cdot 9}{4 \cdot 9} = \frac{27}{36}$	$\quad = \quad$		$\frac{ar}{br}$
"Think of another number."	5	8	10	t
"Multiply the numerator and denominator of the fraction you thought of first by that number."	$\frac{3 \cdot 5}{\quad} = \quad$			$\frac{at}{bt}$
"Your two results are equal!"	$\frac{27}{36} \stackrel{?}{=} \quad$	$\quad \stackrel{?}{=} \quad$	$\quad \stackrel{?}{=} \quad$	$\frac{ar}{br} = \frac{at}{bt}$

Once upon a time, a mathematician stated that there is only one way in which a number can be indicated as a product of the greatest number of factors other than 1 — if the order in which the factors are written is not considered.

For example:

$12 = 4 \cdot 3 = 2 \cdot 6 = 2 \cdot 2 \cdot 3$ and

there is no other group of three factors (other than one) that equals 12.

$30 = 5 \cdot __ = 15 \cdot __ = __ \cdot __ \cdot __$

$16 = 4 \cdot __ = __ \cdot 8 = __ \cdot __ \cdot __ \cdot __$

Is there another way to write 16 as a product of four factors (1 excluded)?

Write each of the following as the product of the largest number of factors other than 1.

a. $8 = __ \cdot __ \cdot __$

b. $24 = __ \cdot __ \cdot __ \cdot __$

c. $27 = __ \cdot __ \cdot __$

d. $36 = __ \cdot __ \cdot __ \cdot __$

e. $40 = __ \cdot __ \cdot __ \cdot __$

f. $21 = __ \cdot __$

g. $18 = __ \cdot __ \cdot __$

h. $60 = __ \cdot __ \cdot __ \cdot __$

i. $78 = __ \cdot __ \cdot __$

j. $22 = __ \cdot __$

k. $48 = __ \cdot __ \cdot __ \cdot __ \cdot __$

l. $54 = __ \cdot __ \cdot __ \cdot __$

m. $68 = __ \cdot __ \cdot __$

n. $95 = __ \cdot __$

o. $100 = __ \cdot __ \cdot __ \cdot __$

List each of the factors indicated in the preceding examples — in order from the smallest to the largest.

2, 3, __, __, __, __, __, __

None of these can be written as the product of two numbers if 1 is excluded. They are called "prime numbers" for this reason. If a factor is a "prime number" it is called a "prime factor".

Write each of the following as the product of its prime factors.

p. $10 =$

q. $32 =$

r. $120 =$

s. $360 =$

t. $64 =$

u. $90 =$

v. $20 =$

w. $25 =$

x. $50 =$

y. $1{,}000 =$

z $128 =$

In any of the examples on this page can you find another set of prime factors for any of the numbers indicated?

Do you believe the mathematician was correct in his statement?

What is the product of all the prime factors that are less than 10? (No factor is to be repeated.)

$__ \cdot __ \cdot __ \cdot __ = ____$

Write each numerator and each denominator in terms of its prime factor.

a $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot}$

b $\frac{15}{20} = \frac{3 \cdot}{\cdot \ \cdot}$

c $\frac{12}{27} =$ ——

d $\frac{30}{77} =$ ——

e $\frac{10}{21} =$ ——

f $\frac{52}{105} =$ ——

Follow the same procedure in the next examples. Then rewrite, arranging all common factors together on one side or put parentheses around them.

g $\frac{30}{45} = \frac{2 \cdot \ \cdot}{5 \cdot} = \frac{2 \cdot (3 \cdot 5)}{\cdot (3 \cdot 5)}$

h $\frac{60}{70} =$ —— $= \frac{(\ \cdot\ )}{(\ \cdot\ )}$

i $\frac{18}{45} =$ —— $=$ ——

j $\frac{66}{72} =$ —— $=$ ——

Follow procedure as in examples above. Then take additional steps as indicated.

k $\frac{90}{165} = \frac{2 \cdot 3 \cdot 3 \cdot}{\cdot \ \cdot (\ \cdot\ )} = \frac{2 \cdot 3}{11} \cdot \frac{3 \cdot 5}{3 \cdot 5} = \frac{6}{} \cdot \frac{15}{} = \frac{6}{11} \cdot 1 =$ ——

l $\frac{45}{70} =$ —— $=$ —— $=$ —— $\cdot$ —— $=$ —— —— $=$ —— $\cdot\ 1 =$ ——

m $\frac{27}{60} =$ —— $=$ —— $=$ —— $\cdot$ —— $=$ —— $\cdot$ —— $=$ —— $\cdot\ 1 =$ ——

n $\frac{44}{56} =$ —— $=$ —— $=$ —— $\cdot$ —— $=$ —— $\cdot$ —— $=$ —— $\cdot\ 1 =$ ——

o $\frac{75}{120} =$ —— $=$ —— $=$ —— $\cdot$ —— $=$ —— $\cdot$ —— $=$ —— $\cdot\ 1 =$ ——

In the next examples, follow the same procedure but record only the results that would fall in the blue-tinted spaces above.

p $\frac{12}{44} = \frac{3}{} \cdot \frac{4}{4} = \frac{3}{}$

q $\frac{36}{42} =$ —— $\cdot$ —— $=$ ——

r $\frac{48}{64} =$ —— $\cdot$ —— $=$ ——

s $\frac{34}{51} =$ —— $\cdot$ —— $=$ ——

t $\frac{28}{70} =$ —— $\cdot$ —— $=$ ——

u $\frac{60}{72} =$ —— $\cdot$ —— $=$ ——

v $\frac{48}{108} =$ —— $\cdot$ —— $=$ ——

w $\frac{57}{76} =$ —— $\cdot$ —— $=$ ——

x $\frac{ak}{bk} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b}$

Mike's Investigation

After much experimenting on his own, Mike presented the following argument to his class.

"Consider any true statement about a product and its factor, such as the following." (Complete each statement so that it is true.)

A	B	C
$3 \cdot 4 =$	$8\overline{)24}$	$\frac{42}{6} =$
D	E	F
$\frac{1}{2} \cdot 7 =$	$3 \div \frac{1}{3} =$ ▢	▢ $\overline{)72}$
G	H	I
$\frac{4}{3} =$	$\frac{1}{2} \div \frac{1}{2} =$	$\frac{2}{3} \cdot \frac{2}{3} =$

"Now," Mike continued, "select any number you like. In each example, multiply the product and either of the factors by that number; and write out the results. Suppose you think of 2, and that the factor you select is indicated by the green tints above.

a	b	c
$6 \cdot 4 = 24$	$16\overline{)\quad}$	$\frac{\quad}{\quad} =$
d	e	f
$\cdot 7 =$	$6 \div$	$\overline{)\quad}$
g	h	i
$\frac{\quad}{\quad} = 1\frac{1}{3}$		$\frac{2}{3} \cdot$

"If you had selected the other factor and multiplied it and the product by 2, the results are:

a	b	c
$3 \cdot 8 =$	$8\overline{)6}$	$\frac{\quad}{6} =$
d	e	f
$\frac{1}{2} \cdot \quad =$		
g	h	i
$\frac{\quad}{3} =$	$\div \frac{1}{2} =$	

"But there's nothing particular about choosing 2 for the experiment," Mike explained, "so I tried 5, multiplying the product and the factor indicated by the green tint, each by 5. My results were as follows.

a	b	c
$15 \cdot 4 =$	$3\overline{)\quad}$	$\frac{\quad}{\quad} =$
d	e	f
$\frac{5}{2} \cdot 7 =$	$\div \frac{1}{3} =$	$6\overline{)\quad}$
g	h	i
$= 1\frac{1}{3}$	$= 1$	$\frac{2}{3} \cdot \frac{\quad}{\quad} =$

"Next," Mike explained, "I thought of a fraction — $\frac{1}{2}$ — and extended the experiment, multiplying the product and the factor indicated by the green tint by $\frac{1}{2}$.

a	b	c
$\frac{3}{2} \cdot 4 = 6$	$3\overline{)\quad}$	
d	e	f
g	h	i

"I point to the patterns I have been investigating in the following shorthand statements. I use 'n' to indicate 'any number I think of.'"

IF
$a \cdot b = c$
then
$an \cdot b = cn$
and
$a \cdot bn = cn$

IF
$c \div a = b$
then
$cn \div an = b$
and
$cn \div a = bn$

Do you agree with Mike?

No one could find a single example in which Mike's pattern led to any confusing results. But some wondered if the idea would be useful at all.

"Yes," Mike assured them. "One use I make is to change a division problem into an example of multiplication.

"Consider the question: How much is $6 \div \frac{1}{3}$? I think of 3 or $\frac{3}{1}$. My experiments led to the conclusion that

$$6 \div \frac{1}{3} = 6 \cdot \frac{3}{1} \div \frac{1}{3} \cdot \frac{3}{1}$$

"I chose a number — 3 or $\frac{3}{1}$ — with a purpose in mind. I knew that $\frac{1}{3} \cdot \frac{3}{1}$ equals 1. Now, I know that

$$6 \div \frac{1}{3} = 6 \cdot 3 \div 1 = 6 \cdot 3 = 18$$

"The trick here depends on the fact that any number divided by 1 is unchanged:

$$3 \div 1 = 3, \quad \frac{1}{2} \div 1 = \frac{1}{2}, \quad \frac{2}{3} \div 1 = \frac{2}{3},$$

$$8.16 \div 1 = 8.16 \ \ldots \text{etc.}$$

"Next consider a more complicated example such as $\frac{7}{8} \div \frac{2}{3} = ?$ The number I think of is $\frac{3}{2}$. The results are

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} \div \frac{2}{3} \cdot \frac{3}{2} = \frac{7}{8} \cdot \frac{3}{2} \div 1 = \frac{21}{16} = 1\frac{5}{16}$$

"No matter what number the known factor is, I think of a number which, when multiplied by the known factors, gives 1 as the result.

$7 \cdot \frac{1}{7} = 1$ $\quad 11 \cdot __ = 1$ $\quad \frac{1}{2} \cdot __ = 1$

$\frac{2}{3} \cdot __ = 1$ $\quad \frac{5}{4} \cdot __ = 1$ $\quad 1\frac{1}{4} \cdot __ = 1$

$87 \cdot __ = 1$ $\quad \frac{7}{31} \cdot __ = 1$ $\quad 2\frac{1}{2} \cdot __ = 1$

"The pattern I follow can be indicated in shorthand. Since every number I know can be written in the form of a fraction, $\frac{a}{b}$ stands for any number I think of. Then

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$$

"Here are more examples in which I use this idea:"

a. $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \cdot __ \div __ \cdot __ = __ =$

b. $1\frac{1}{8} \div \frac{3}{8} = \frac{9}{8} \cdot __ \div \frac{3}{8} \cdot __ = __ =$

c. $2 \div 3 = \frac{2}{1} \cdot __ \div \frac{3}{1} \cdot __ = __$

d. $\frac{1}{2} \div 2 = \frac{1}{2} \cdot __ \div \frac{2}{1} \cdot __ = __$

e. $\frac{3}{8} \div \frac{3}{4} = __ \cdot __ \div __ \cdot __ = __ = __$

Are these results reasonable?

Think of each example as a headline. Can you make up several stories that fit each headline?

Write headlines for the following:

Mary cut $\frac{2}{3}$ of a yard of ribbon into pieces $\frac{1}{12}$ of a yard long. She had ____ pieces.

How many quarters are there in three and one-half dollars? _____

A cricket could jump for two-thirds of a mile before getting tired. How many stretches of $\frac{2}{3}$ miles would he hop in a straight line to get $6\frac{1}{2}$ miles from home?

_____

Mr. Bowen bought 100 ft. of fence to protect his garden. He wanted a garden that was rectangular in shape and as large as he could enclose with 100 ft. of fencing.

He made his garden the shape of the green outline shown here — 30 feet long and 20 feet wide. Did he use all his fencing?

After he was finished, his neighbor said, "You could have made your garden larger if you had used other dimensions."

"But I had only 100 feet of fencing," Mr. Bowen answered, "and I wanted all the corners to be square."

"I still think I could enclose a larger area," the neighbor replied. Could he have? Mr. Bowen made the following study:

length in feet of each of 2 opposite sides	30	35	40	45	25	23	21	27	28
length in feet of each of other 2 sides	20	15	10	5	25				
square feet in area enclosed	600								

He selected the three best examples and extended his study

length in feet of each of one pair of sides	23	24	24.5	24.8	25	25.1	25.3	25.7	27
length in feet of each of the other pair	27				25				23
number of square feet in enclosed area									

This result interested Mr. Bowen, and he explored the idea. Before beginning, he worked out some shorthand to help him simplify his work.

A stands for number of square ft. in area
l stands for number of ft. in length
w stands for number of ft. in width

$l \cdot w = A$

"Suppose I had 200 feet of fencing?"

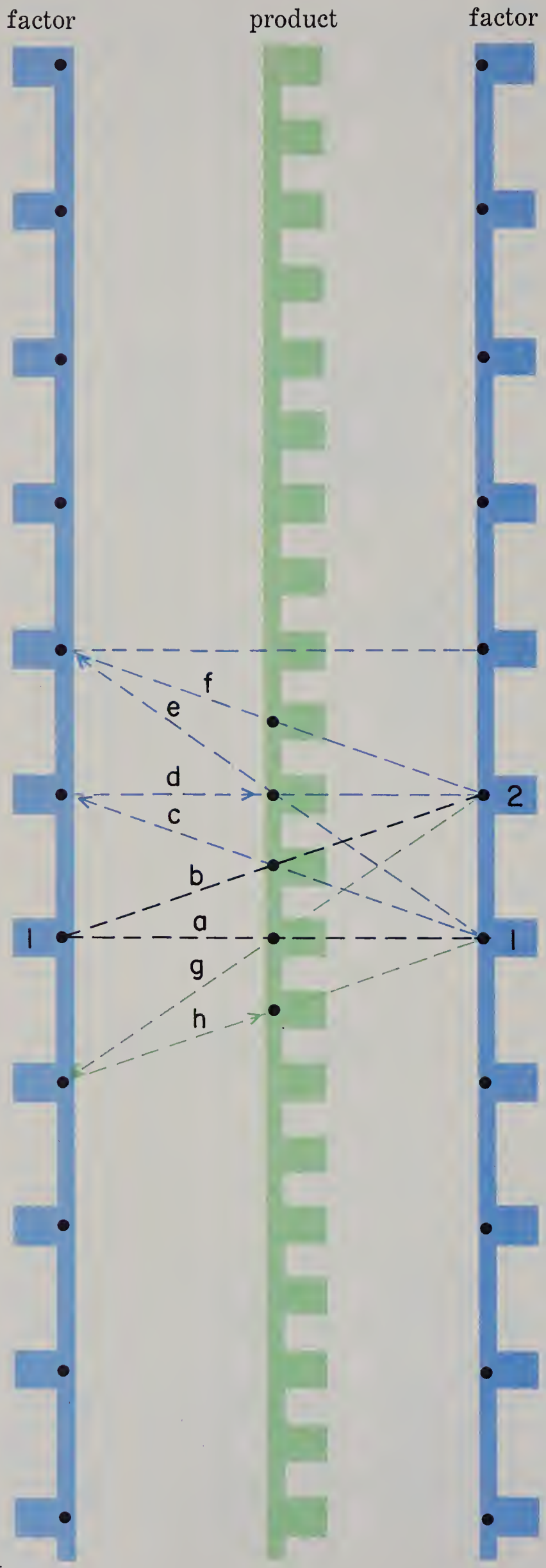
l	25	35	45	50	55	53	51	51.5
w	75							
A								

"Suppose I had only 90 feet of fencing?"

								?
l	40	30	25	20	24	22	23	
w								
A								

Mr. Bowen still was not thoroughly convinced. He tried similar studies, considering the problem each time with a different amount of fencing.

What do you think he decided when he considered having 121 feet? The largest enclosure would be _____ feet long and _____ feet wide. The area enclosed would thus be _____ square feet.



You can build a very limited but useful multiplying machine out of the diagram on the left.

Only three labels are given: 1, 1, and 2.

The plan is that the numbers noted on the blue scales are factors. Where a line connecting two factors crosses the green line, draw a dot and label it to indicate the product.

Lines a and b at the left ask that you complete the following statements:

a. 1 = ▢ ÷ 1

b. 1 = ▢ ÷ 2

The point at which line a crosses the green line will be labeled 1; the point at which b crosses will be labeled 2.

Other lines on the left ask that you complete the following statements:

c. ▢ = 2 ÷ 1

d. 2 = ▢ ÷ 2

e. ▢ = 4 ÷ 1

f. 4 = ▢ ÷ 2

Keep following the same method to find and label all points on the factor lines and the product lines, above line a.

Line g at the left asks that you complete the following statement — to find the missing factor as the label for the first dot on the left below the one labeled 1.

g. ▢ = 1 ÷ 2

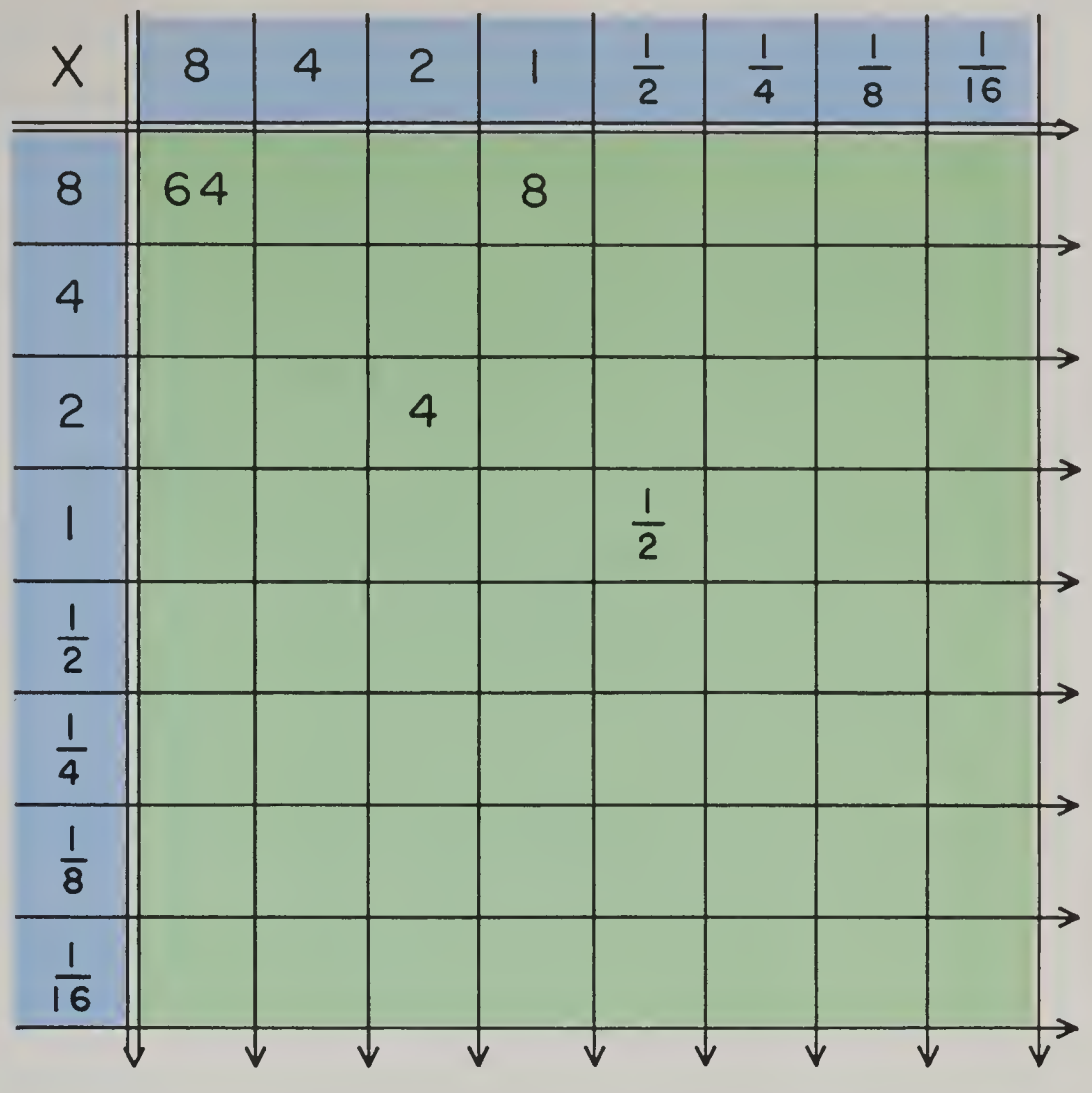
Now line h at the left asks the following:

h. $\frac{1}{2}$ = ▢ ÷ 1

Keep following the same method to find labels for points in the lower part of the diagram.

Use the multiplying machine you have built to fill in the products in the green space in the diagram at the right.

Select a factor from each of the group of numbers indicated in the blue spaces. Find them on the blue lines on the previous page. Connect them with a straight-edge. Note the indicated product.



Find the missing factor.

A. 4 ÷ 8 = ☐

B. 2 ÷ 16 = ☐

C. 2 ÷ $\frac{1}{2}$ = ☐

D. 2 ÷ $\frac{1}{4}$ = ☐

E. 1 ÷ $\frac{1}{2}$ = ☐

F. 1 ÷ $\frac{1}{4}$ = ☐

G. $\frac{1}{2}$ ÷ $\frac{1}{2}$ = ☐

H. $\frac{1}{2}$ ÷ $\frac{1}{4}$ = ☐

I. $\frac{1}{4}$ ÷ $\frac{1}{2}$ = ☐

J. $\frac{1}{8}$ ÷ $\frac{1}{16}$ = ☐

K. $\frac{1}{8}$ ÷ $\frac{1}{8}$ = ☐

L. $\frac{1}{8}$ ÷ $\frac{1}{4}$ = ☐

M. $\frac{1}{8}$ ÷ $\frac{1}{2}$ = ☐

N. $\frac{1}{64}$ ÷ $\frac{1}{16}$ = ☐

O. $\frac{1}{128}$ ÷ $\frac{1}{8}$ = ☐

Write appropriate examples that fit the headlines below:

P. 8 ÷ $\frac{1}{2}$ = ☐

Q. $\frac{1}{2}$ ÷ ☐ = 2

R. ☐ ÷ 4 = $\frac{1}{8}$

S. $\frac{1}{4}$ ÷ 2 = ☐

T. $\frac{1}{16}$ ÷ $\frac{1}{8}$ = ☐

It may seem strange to you, but for thousands of years no one thought of writing fractions with any numerals other than 1. (There was one exception: they had a special symbol for two-thirds.) Neither, in olden times, did they repeat denominators. They did not write $\frac{2}{5}$ as $\frac{1}{5}$ plus $\frac{1}{5}$, for example. Rather, they would write it as the sum of $\frac{1}{4}$, $\frac{1}{10}$ and $\frac{1}{20}$.

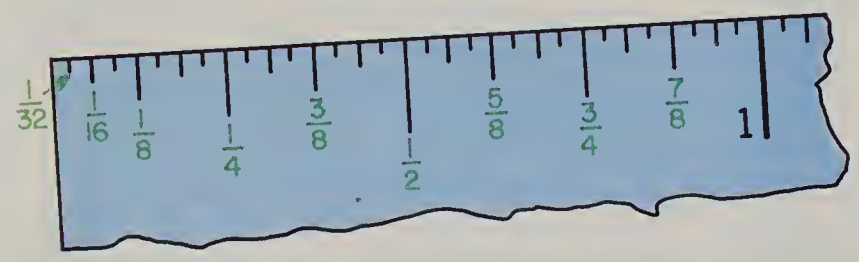
Try to think as men and women thought 3500 years ago. And remember that no one had invented the sign for addition (+), so they wrote this way:

$\frac{3}{4}$ was written $\frac{1}{2}, \frac{1}{4}$

$\frac{5}{16}$ was written $\frac{1}{4}, \frac{1}{16}$

$\frac{19}{32}$ was written $\frac{1}{2}, \frac{1}{16}, \frac{1}{32}$

If you think about the division on a ruler, you won't have much trouble with the examples above and below. Through a magnifying glass, an inch on a ruler looks like this:



$\frac{5}{8}$	$\frac{1}{2}, \frac{\quad}{8}$
$\frac{3}{8}$	$\frac{\quad}{1}, \frac{\quad}{1}$
$\frac{9}{16}$	$\frac{1}{2}, \frac{\quad}{1}$
$\frac{\quad}{\quad}$	$\frac{1}{2}, \frac{1}{4}$
$\frac{7}{8}$	$\frac{1}{2}, \frac{\quad}{1}, \frac{\quad}{1}$
$\frac{3}{16}$	$\frac{\quad}{1}, \frac{\quad}{1}$
$\frac{7}{16}$	$\frac{\quad}{1}, \frac{\quad}{1}, \frac{1}{16}$
$\frac{9}{16}$	$\frac{\quad}{1}, \frac{\quad}{1}$
$\frac{11}{16}$	$\frac{\quad}{1}, \frac{\quad}{1}, \frac{\quad}{1}$

$\frac{13}{16}$	$\frac{\quad}{1}, \frac{\quad}{1}, \frac{\quad}{1}$
$\frac{3}{32}$	$\frac{\quad}{1}, \frac{\quad}{1}$
$\frac{5}{32}$	$\frac{\quad}{\quad}, \frac{\quad}{\quad}$
$\frac{7}{32}$	$\frac{\quad}{1}, \frac{\quad}{1}, \frac{\quad}{1}$
$\frac{13}{32}$	$\frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}$
$\frac{15}{16}$	$\frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}$
$\frac{15}{32}$	$\frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}$
$\frac{21}{32}$	$\frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}$
$\frac{27}{32}$	$\frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}, \frac{\quad}{\quad}$

Circle those fractions below which can be reduced to unit fractions — with 1 as numerator.

$\frac{3}{6}$ $\frac{6}{9}$ $\frac{6}{8}$ $\frac{4}{12}$ $\frac{6}{12}$ $\frac{8}{12}$ $\frac{2}{10}$ $\frac{4}{10}$

$\frac{12}{16}$ $\frac{6}{15}$ $\frac{25}{100}$ $\frac{22}{33}$ $\frac{7}{28}$ $\frac{9}{99}$ $\frac{13}{52}$

In the top rows of each chart below, complete the fractions so they are already or can be reduced to unit fractions. Indicate the unit fraction in the bottom row.

I

$\frac{1}{6}$	$\frac{2}{6}$	$\frac{\quad}{6}$	$\frac{\quad}{8}$	$\frac{\quad}{8}$	$\frac{\quad}{8}$	$\frac{\quad}{9}$	$\frac{\quad}{9}$	$\frac{\quad}{10}$
$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{\quad}{1}$	$\frac{\quad}{1}$	$\frac{\quad}{1}$	$\frac{\quad}{1}$	$\frac{\quad}{1}$	$\frac{\quad}{1}$

II

$\frac{\quad}{10}$	$\frac{\quad}{10}$	$\frac{\quad}{12}$	$\frac{\quad}{12}$	$\frac{\quad}{12}$	$\frac{\quad}{12}$	$\frac{\quad}{12}$	$\frac{\quad}{15}$	$\frac{\quad}{15}$
$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$

III

$\frac{\quad}{15}$	$\frac{\quad}{18}$	$\frac{\quad}{18}$	$\frac{\quad}{18}$	$\frac{\quad}{18}$	$\frac{\quad}{18}$	$\frac{\quad}{20}$	$\frac{\quad}{20}$	$\frac{\quad}{20}$
$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$

IV

$\frac{\quad}{24}$	$\frac{\quad}{24}$	$\frac{\quad}{24}$	$\frac{\quad}{24}$	$\frac{\quad}{24}$	$\frac{\quad}{24}$	$\frac{\quad}{24}$	$\frac{\quad}{20}$	$\frac{\quad}{20}$
$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$	$\frac{1}{\quad}$

How can we write $\frac{7}{12}$ as sums of unit fractions? Look in row II above.

$\frac{7}{12} = \frac{6}{12} + \frac{1}{12}$ or $\frac{4}{12} + \frac{3}{12}$ or $\frac{4}{12} + \frac{2}{12} + \frac{1}{12}$

Now reduce all fractions to unit fractions:

$\frac{7}{12} = \frac{1}{2} + \frac{1}{\quad}$ or $\frac{1}{3} + \frac{1}{\quad}$ or $\frac{1}{3} + \frac{1}{\quad} + \frac{1}{\quad}$

Find two ways to write $\frac{11}{12}$ as the sum of three unit fractions:

$\frac{11}{12} = \frac{\quad}{\quad} + \frac{\quad}{\quad} + \frac{\quad}{\quad}$ or $\frac{\quad}{\quad} + \frac{\quad}{\quad} + \frac{\quad}{\quad}$

What is the largest fraction that can be expressed as the sum of 2 different unit fractions?

$$\frac{\quad}{\quad} = \frac{1}{\quad} + \frac{1}{\quad}$$

Is there more than one way to express that fraction as the sum of 2 different unit fractions?

Make each of the following into fractions that are less than 1 and whose numerators are more than 1.

$\frac{3}{}$ $\frac{11}{}$ $\frac{5}{}$ $\frac{13}{}$ $\frac{19}{}$ $\frac{29}{}$ $\frac{7}{}$ $\frac{17}{}$ $\frac{31}{}$

$\frac{47}{}$ $\frac{61}{}$ $\frac{71}{}$ $\frac{37}{}$ $\frac{59}{}$ $\frac{73}{}$ $\frac{67}{}$ $\frac{43}{}$ $\frac{53}{}$

How many can be reduced?

How many could be reduced if you had selected other numerators that meet the rules given above?

Can you use any denominator less than 75 which behaves in the same way — and has not been used in one of the examples above?

Arrange these and the numbers used above as denominators in order of size beginning with 3, 5, etc.

$\frac{3}{}$ $\frac{5}{}$ $\frac{7}{}$ — — — — — — —

— — — — — — — — — —

How many other numbers that behave similarly can you find that are less than 100?

— — — — — —

Every whole number that is evenly divisible by only two different numbers — itself and 1 — is called a PRIME NUMBER.

Is 1 a prime number?

Is 2 a prime number?

Is there any other even number that is a prime number?

Write fractions below in which:

a. numerators are prime numbers

— — — — — —

b. denominators are prime numbers

— — — — — —

c. numerators and denominators are both prime numbers.

— — — — — —

Could you follow the directions above at c., and write a fraction that could be reduced?

Can you write $\frac{2}{7}$ as the sum of unit fractions?

Yes $\frac{2}{7} = \frac{4}{14} = \frac{6}{21} = \frac{8}{28}$

Write out all fractions with 28 as a denominator that can be reduced to unit fractions.

	$\frac{14}{28}$	$\frac{7}{28}$	$\frac{}{28}$	$\frac{}{28}$	$\frac{}{28}$	
Unit fractions	$\frac{1}{2}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	

$\frac{2}{7} = \frac{8}{28} = \frac{7}{28} + \frac{1}{28} = \frac{1}{4} + \frac{1}{28}$

$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$

We also know that $\frac{2}{7} = \frac{12}{42}$. Now complete the following so they can be reduced to unit fractions:

	$\frac{21}{42}$	$\frac{14}{42}$	$\frac{}{42}$	$\frac{}{42}$	$\frac{}{42}$	$\frac{}{42}$	$\frac{}{42}$
Unit fractions	$\frac{1}{2}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$	$\frac{1}{}$

$\frac{2}{7} = \frac{12}{42} = \frac{}{42} + \frac{}{42} + \frac{}{42} = \frac{1}{} + \frac{1}{} + \frac{1}{}$

UNIT FRACTIONS

How many fractions less than one and with denominators less than 100 are, or can be, reduced to unit fractions? _ _ _ _ _

1/2	1/3	2/4	1/4	1/5	3/6	2/6	1/6	7	8	8	8	9	9	10	10	10	11
12	12	12	12	12	13	14	14	14	15	15	15	16	16	16	16	17	18
18	18	18	18	19	20	20	20	20	20	21	21	21	22	22	22	23	24
24	24	24	24	24	24	25	25	26	26	26	27	27	27	28	28	28	28
28	29	30	30	30	30	30	30	30	31	32	32	32	32	32	33	33	33
34	34	34	35	35	35	36	36	36	36	36	36	36	36	37	38	38	38
39	39	39	40	40	40	40	40	40	40	41	42	42	42	42	42	42	42
43	44	44	44	44	44	45	45	45	45	45	46	46	46	47	48	48	48
48	48	48	48	48	48	49	49	50	50	50	50	50	51	51	51	52	52
52	52	52	53	54	54	54	54	54	54	54	55	55	55	56	56	56	56
56	56	56	57	57	57	58	58	58	59	60	60	60	60	60	60	60	60
60	60	60	61	62	62	62	63	63	63	63	63	64	64	64	64	64	64
65	65	65	66	66	66	66	66	66	66	67	68	68	68	68	68	69	69
69	70	70	70	70	70	70	70	71	72	72	72	72	72	72	72	72	72
72	72	73	74	74	74	75	75	75	75	75	76	76	76	76	76	77	77
77	78	78	78	78	78	78	78	79	80	80	80	80	80	80	80	80	80
81	81	81	81	82	82	82	83	84	84	84	84	84	84	84	84	84	84
84	85	85	85	86	86	86	87	87	87	88	88	88	88	88	88	88	89
90	90	90	90	90	90	90	90	90	90	90	91	92	92	92	92	92	93
93	93	94	94	94	95	95	95	96	96	96	96	96	96	96	96	96	96
96	97	98	98	98	98	98	99	99	99	99	99						

If you select any three numbers as addends, in how many ways can you fill in the expression __ + __ + __ , if order is considered?

Group of three addends selected ↓

Are all expressions equal? Let's investigate!

7 11 6	7 + 11 + 6 = 7 + 6 + 11 = 6 + 11 + 7 = 6 + 7 + 11 = 11 + 6 + 7 = 11 + 7 + 6 18 + 6 = 13 + 11 = 17 + 7 = __ + __ = __ + __ = __ + __ 24 = 24 = 24 = __ = __ = __
9 12 15	9 + 12 + 15 = __ + __ + __ = __ + __ + __ = __ + __ + __ = __ + __ + __ = __ + __ + __ 21 + 15 = __ + __ = __ + __ = __ + __ = __ + __ = __ + __ 36 = __ = __ = __ = __ = __
16 26 18	16 + 26 + 18 = __ + __ + __ = __ + __ + __ = __ + __ + __ = __ + __ + __ = __ + __ + __ 42 + 18 = __ + __ = __ + __ = __ + __ = __ + __ = __ + __ __ = __ = __ = __ = __ = __
a b c	One way to express this idea is: a + b + c = a + c + b = b + a + c = b + __ + __ = __ + __ + __ = __ + __ + __

Another way to give examples and indicate the idea is:

58 + 47 + 80 = 58 + 47 + __
105 + 80 = 58 + __
185 = __

125 + 76 + 104 = __ + 76 + 104
__ + __ = __ + __
__ = __

85 + 99 + 106 = 85 + __ + 106
__ + __ = __ + __
__ = __

$(x + y) + z = x + (y + z)$

39 + 87 + 64 = 39 + 87 + __
__ + __ = __ + __
__ = __

$r + (s + t) = (r + s) + t$

A similar study using groups of three facts follows:

3 · 5 · 8 = 3 · 5 · __
__ · __ = __ · __
__ = __

6 · 7 · 9 = 6 · __ · 9
__ · __ = __ · __
__ = __

__ · 4 · 12 = 11 · 4 · 12
__ · __ = __ · __
__ = __

$(a \cdot b)c = a(b \cdot c)$

10 · 15 · 12 = 10 · 15 · __
__ · __ = __ · __
__ = __

$k(l \cdot m) = (k \cdot l)m$

Do you believe that the following patterns will always give correct results regardless of the groups of numbers chosen as addends or factors in each case?

$(3+4)+(5+6)=(3+5)+(4+6)$

$7 + 11 = __ + __$

$18 = __$

$(3\cdot 4)\cdot(5\cdot 6)=(3\cdot 5)\cdot(4\cdot 6)$

$12 \cdot __ = __ \cdot __$

$___ = ____$

$(2+7)+(10+12)=(2+10)+(7+12)$

$__ + __ = __ + __$

$__ = __$

$(2\cdot 7)\cdot(10\cdot 12)=(2\cdot 10)\cdot(7\cdot 12)$

$__ \cdot __ = __ \cdot __$

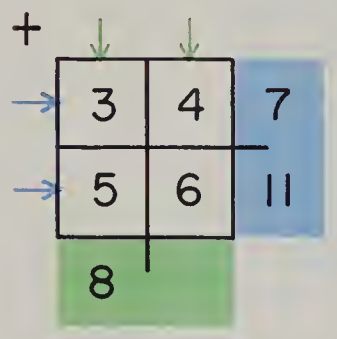
$____ = ____$

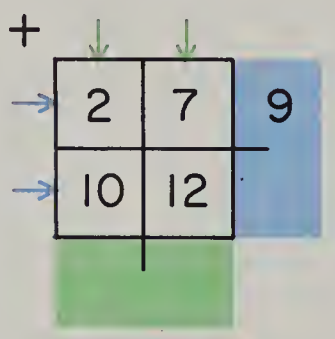
$(a+b)+(c+d)=(a+c)+(b+d)$ and $(a\cdot b)\cdot(c\cdot d)=(a\cdot c)\cdot(b\cdot d)$

Do you believe these patterns suggest a reliable idea?

CROSS NUMBER PUZZLES

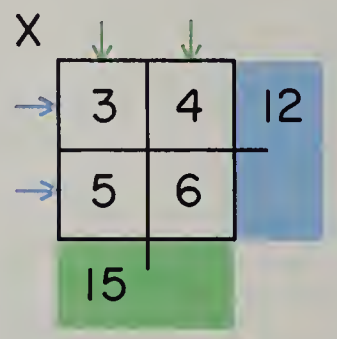
The rules are simple for the "addition" puzzles. In a grid of four blocks like the one below, the sums of the two numbers in each row are noted; and then the sums of the two numbers in each column are noted.

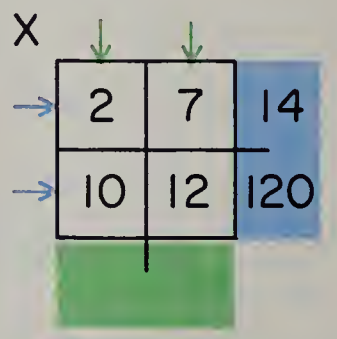




You have solved the puzzle if, at this point, the sum of the numbers in the blue tint is the same as the sum of numbers in the green tint. Is it true that $8 + 10 = 7 + 11$? and $12 + 19 = 9 + 22$?

A "multiplication" puzzle follows the same basic pattern, except that the numbers written are considered as factors. Products of rows and columns are written in the tinted areas.

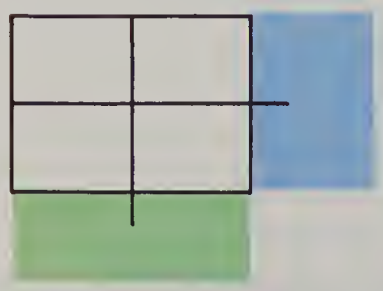


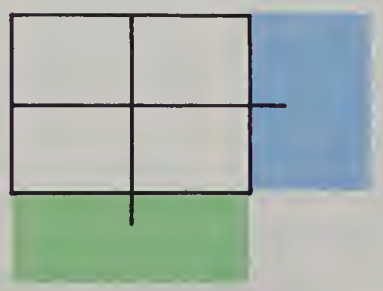


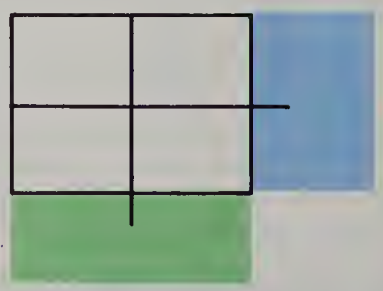
You have solved the puzzle if, at this point, the product of the numbers in the blue tint is the same as the product of the numbers in the green tint. Is it true that $15 \cdot 24 = 12 \times 30$? and $20 \times 84 = 14 \cdot 120$?

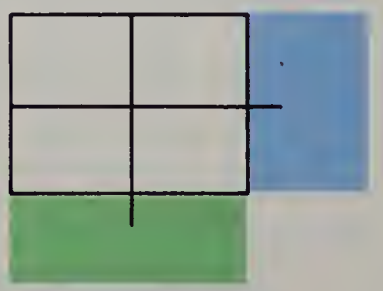
If you have solved these puzzles, note the final sum or product in the lower right-hand corner.

Try some on your own:









Can you find any solutions?

Did you find any solutions to the Cross Number Puzzles on the previous page? If so, now try to find a group of four numbers that is not a solution; this is a more difficult task.

Is it impossible?

Let's use a, b, c and d to suggest the idea of any group of four numbers thought of. The rule is addition.

+

a	b	$a+b$
c	d	$c+d$

Does $(a+b)+(c+d)=(a+c)+(b+d)$?

Try the rule for multiplication:

×

a	b	$a \cdot b$
c	d	

Does $(a \cdot b) \cdot (c \cdot d)=(a \cdot c) \cdot (b \cdot d)$?

Work through this collection of Cross Number Puzzles (some of the 57 varieties). Watch for the sign in the upper left-hand corner that tells you which rule is to be followed.

+

$\frac{1}{8}$	$\frac{3}{8}$	–
$\frac{1}{8}$	$\frac{1}{4}$	–
–	–	$\frac{7}{8}$

+

123	456	
777	777	

×

	4	
6		48
12		

+

36	45	
27	54	
18	63	
9	72	

+

5	6	7	8	9	10	11	12	
4	3	2	1	16	15	14	13	

×

$\frac{1}{2}$	$\frac{3}{4}$	
$\frac{3}{5}$	$\frac{7}{11}$	

×

$\frac{2}{3}$	$\frac{4}{5}$	
		$\frac{2}{15}$
$\frac{4}{15}$		

×

$\frac{1}{7}$		$\frac{1}{28}$
$\frac{2}{21}$	$\frac{3}{16}$	

×

	$\frac{3}{4}$	$\frac{1}{2}$
$\frac{4}{5}$		
$\frac{8}{15}$		$\frac{1}{3}$

+

100	70	9	
600	80	5	

+

300		6	356
			484
700	130		

+

			707
	90	5	295
900	90		

+

400		5	475
800	70		883

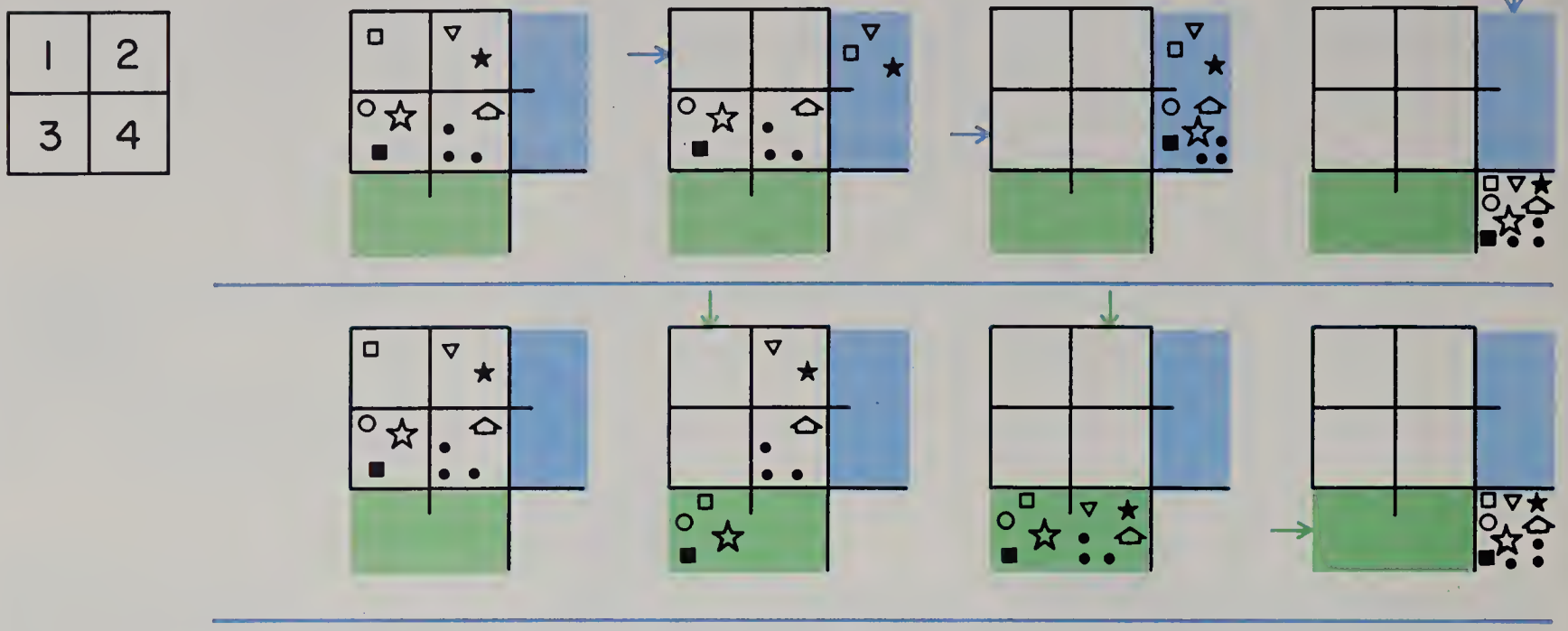
+

	80	9	569
215	65		
			857

+

600		8	829
350		4	592

Design your own Cross Number Puzzles. Invent your own new varieties.



The diagram above is a powerful argument that the "addition" variety of Cross Number Puzzles is bound to work every time.

Think of any four numbers. Note them in a square of four blocks. Copy this pattern using objects — as many objects in each block as the numbers indicate.

Sweep the top row of objects to the right; sweep the bottom row of objects to the right — into the blue space. Now sweep the objects out of the blue space into the bottom right corner. Count them.

Replace the objects as they were.

This time, sweep down the left-hand column; then down the next column — into the green space. Now sweep across into the bottom right-hand corner. Count the objects.

Your two counts were the same! If they hadn't been, what would you think? That you counted wrong? That you lost something along the way? At any rate, no one could convince you that the Cross Number Puzzle doesn't always work, could they?

We have expressed this idea in this way:

$$(a+b)+(c+d)=(a+c)+(\ \ +\ \ )$$

Here is another example of this pattern:

$$(60+4)+(20+8)=(60+20)+(4+8)$$

$$64 + 28 = 80 + 12$$

$$\begin{array}{r} 64 \\ +28 \\ \hline \end{array} = \begin{array}{r} 60+4 \\ +20+8 \\ \hline \end{array}$$

Do you recognize the pattern?

$$\begin{array}{r} 47 \\ +36 \\ \hline \end{array} = \begin{array}{r} 40+7 \\ +30+6 \\ \hline \end{array} = \begin{array}{r} 22+25 \\ +18+18 \\ \hline \end{array}$$

One "subtraction" variety of Cross Number Puzzles follows a simple rule. When working vertically, the number in the second row (blue-tinted) is always subtracted from the one above it. Otherwise, the rules of the "addition" variety are followed.

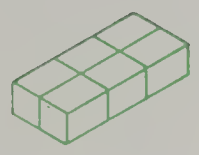
25	8	33
15		
10	2	12

40	15	
20	8	28

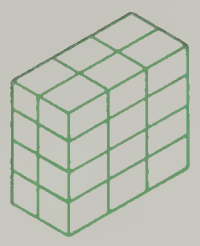
	14	124
	8	78

	8	68
	7	277

If we begin with two rows with three blocks in each row, we can use this to suggest the idea of $2 \cdot 3$. Then we can suggest the idea of $(2 \cdot 3)4$ by piling up four layers of $2 \cdot 3$ blocks.

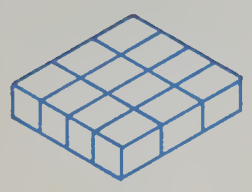


$2 \cdot 3$ blocks

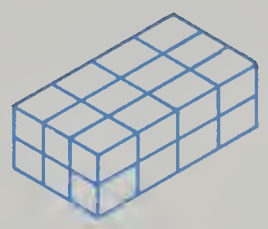


$(2 \cdot 3)4$ blocks

If we begin with three rows of four blocks to suggest the idea of $3 \cdot 4$, then two such layers suggest the idea of $2(3 \cdot 4)$.

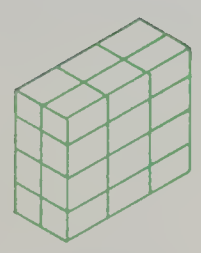


$3 \cdot 4$ blocks

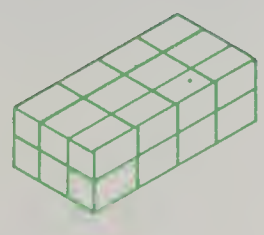


$2(3 \cdot 4)$ blocks

But the green stack of blocks, if tipped on its side, has the same number of blocks as the blue stack.

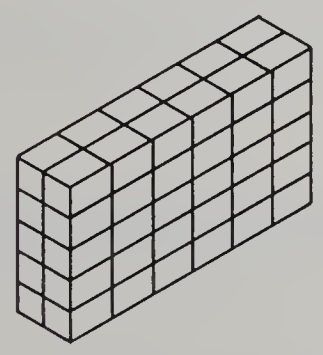


$(2 \cdot 3)4$ =

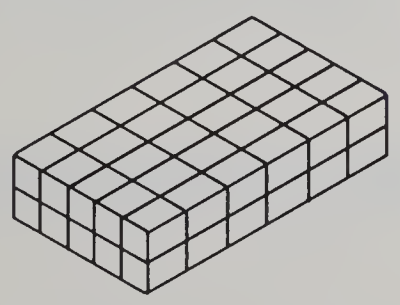


$2(3 \cdot 4)$

The numbers we choose to indicate number of rows, number in each row and number of layers does not affect the outcome.



$(2 \cdot 6)5$ =



$2(6 \cdot 5)$

We can express this idea in this way:

$$(x \cdot y)z = x(y \cdot z) \text{ or } (xy)z = x(yz) \text{ or } x(y \cdot z) = (x \cdot y)z \text{ or } x(yz) = (xy)z$$

If an expression contains addends only, we can change the order of the addends at will without affecting the sum. We call this the

"in-any-order" rule for addition.

If an expression contains factors only, we can change the order of the factors without affecting the product. We call this the

"in-any-order" rule for multiplication.

These rules are very helpful in simplifying computations. All examples below can be done quickly "in your head" if you use the "in-any-order" rules.

$7 + 8 + 2 + 3 = (7 + 3) + (8 + 2) = 10 + 10 = 20$

$51 + 85 + 49 = (51 + 49) + 85 =$

$25 + 79 + 11 + 75 = 100 + 90 =$

$44 + 75 + 6 =$

$125 + 89 + 75 + 1 =$

$450 + 780 + 550 + 220 =$

$38 \cdot 25 \cdot 4 = (25 \cdot 4) \cdot 38 = 100 \cdot 38 = 3{,}800$

$5 \cdot 125 \cdot 2 =$

$2 \cdot 87 \cdot 50 \cdot 10 = (2 \cdot 50 \cdot 10) \cdot 87 =$

$5 \cdot 9 \cdot 4 \cdot 7 \cdot 5 = (5 \cdot 4 \cdot 5)(9 \cdot 7) =$

$6 \cdot 231 \cdot 5 =$

$15 \cdot 9 \cdot 6 =$

$8 + 7 + 3 + 2 + 9 + 6 + 1 + 4 =$

$15 + 17 + 19 + 21 + 23 + 25 =$

$5 \cdot 3 \cdot 20 \cdot 35 =$

$2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 5 \cdot 5 =$

$18 + 79 + 12 + 11 =$

$7 \cdot 8 \cdot 9 \cdot 25 =$

$467 + 269 + 533 + 1{,}721 =$

The "in-any-order" rules apply only when an expression is limited either to addends or to factors — but not when they are combined. Consider:

$$3 + 5 \cdot 9 =$$

Bill argued that $3 + 5 = 8$ and $8 \cdot 9 = 72$, so, "72 is the answer," he concludes.

Mary thought that $5 \cdot 9 = 45$ and $3 \cdot 45 = 135$, so, "135 is the answer," she claimed.

To prevent such arguments we need to agree on another "order" rule:

Unless otherwise indicated, perform all indicated multiplications first.

Without such an agreement, we might often be misunderstood. So:

$$3 + 5 \cdot 9 = 3 + 45 = 135$$

"But," complained Bill, "suppose I mean that $3 + 5 = 8$ and $8 \cdot 9 = 72$. How do I indicate that? We need another agreement, I guess." We could say:

Unless otherwise indicated by parentheses, perform multiplications first.

Then when I mean $3 + 5 = 8$ and $8 \cdot 9 = 72$, I can write:

$$(3 + 5)\,9 = 8 \cdot 9 = 72$$

What about this expression?

$$45 \div 5 \cdot 3$$

Rather than complicating the rules, let's agree to use parentheses or a fractional form to indicate our meaning.

$(45 \div 5)3 = 9 \cdot 3 = 27$ or $\frac{45}{5} \cdot 3 = 9 \cdot 3 = 27$

or

$45 \div (5 \cdot 3) = 45 \div 15 = 3$ or $\frac{45}{5 \cdot 3} = \frac{45}{15} = 3$

With these simple rules, you know exactly what is meant by the following:

$4 + 3 \cdot 10 + 6 =$ ___	$(4 + 3)(10 + 6) =$ ___
$(4 + 3)\,10 + 6 =$ ___	$4 + 3 \cdot 10 \cdot 6 =$ ___
$4 + 3\,(10 + 6) =$ ___	$4 \cdot 3 + 10 \cdot 6 =$ ___

If you discover the pattern in each of the groups below, the examples involve only very simple computations.

$2 \cdot 9 + 3 \cdot 9 =$ ____	$3 \cdot 43 + 2 \cdot 43 =$ ____
$2 \cdot 11 + 3 \cdot 11 =$ ____	$91 \cdot 31 + 9 \cdot 31 =$ ____
$2 \cdot 17 + 3 \cdot 17 =$ ____	$49 \cdot 67 + 51 \cdot 67 =$ ____
$4 \cdot 12 + 6 \cdot 12 =$ ____	$18 \cdot 48 + 2 \cdot 48 =$ ____
$7 \cdot 23 + 3 \cdot 23 =$ ____	$23 \cdot 17 + 27 \cdot 17 =$ ____
$8 \cdot 37 + 2 \cdot 37 =$ ____	$36 \cdot 9 + 64 \cdot 9 =$ ____
$5 \cdot 85 + 5 \cdot 85 =$ ____	$155 \cdot 8 + 45 \cdot 8 =$ ____

$$rs + ts = (r + t)\,s$$

$6 \cdot 2 + 6 \cdot 8 =$ ____	$8 \cdot 85 + 8 \cdot 15 =$ ____
$9 \cdot 17 + 9 \cdot 3 =$ ____	$31 \cdot 15 + 31 \cdot 5 =$ ____
$18 \cdot 5 + 18 \cdot 5 =$ ____	$18 \cdot 91 + 18 \cdot 9 =$ ____
$21 \cdot 4 + 21 \cdot 3 =$ ____	$7 \cdot 25 + 7 \cdot 15 =$ ____
$19 \cdot 3 + 19 \cdot 2 =$ ____	$89 \cdot 78 + 89 \cdot 22 =$ ____
$87 \cdot 2 + 87 \cdot 4 =$ ____	$53 \cdot 51 + 53 \cdot 19 =$ ____
$4 \cdot 18 + 4 \cdot 7 =$ ____	$12 \cdot 42 + 12 \cdot 18 =$ ____

$$mx + my = m\,(x + \quad)$$

$4 \cdot 19 + 19 \cdot 6 =$ ____	$40 \cdot 87 + 87 \cdot 60 =$ ____
$7 \cdot 8 + 8 \cdot 3 =$ ____	$31 \cdot 29 + 29 \cdot 29 =$ ____
$5 \cdot 13 + 13 \cdot 15 =$ ____	$95 \cdot 9 + 9 \cdot 5 =$ ____
$18 \cdot 4 + 4 \cdot 7 =$ ____	$15 \cdot 71 + 71 \cdot 25 =$ ____
$12 \cdot 5 + 5 \cdot 8 =$ ____	$21 \cdot 8 + 8 \cdot 29 =$ ____

$$ab + bc = b\,(a \quad)$$

$9 \cdot 2 + 3 \cdot 9 =$ ____	$8 \cdot 46 + 64 \cdot 8 =$ ____
$4 \cdot 14 + 16 \cdot 4 =$ ____	$99 \cdot 18 + 82 \cdot 99 =$ ____
$17 \cdot 8 + 2 \cdot 17 =$ ____	$64 \cdot 46 + 54 \cdot 64 =$ ____

$$xy + zx = x\,(\quad + \quad)$$

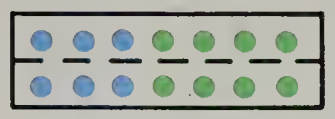
Quick Checking

$4 \cdot 9 + 6 \cdot 9 =$ ____	$14 \cdot 9 + 9 \cdot 6 =$ ____
$15 \cdot 8 + 15 \cdot 2 =$ ____	$29 \cdot 37 + 29 \cdot 63 =$ ____
$17 \cdot 7 + 3 \cdot 17 =$ ____	$40 \cdot 97 + 60 \cdot 97 =$ ____
$24 \cdot 37 + 26 \cdot 37 =$ ____	$18 \cdot 22 + 22 \cdot 32 =$ ____
$91 \cdot 18 + 91 \cdot 2 =$ ____	$94 \cdot 49 + 6 \cdot 49 =$ ____

3 + 4
or 4 + 3

2·3
or 3·2

2·4
or 4·2

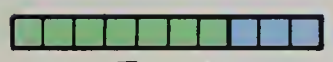


2(3+4)
or 2(4+3)
or (3+4)2
or (4+3)2

2·3 + 2·4
or 2·4 + 2·3
or 3·2 + 4·2
or 4·2 + 4·3

Conclusions:

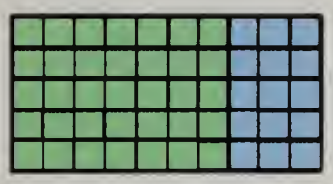
2(3+4) = 2·3 + 2·4
2(4+3) = 2·4 +
(3+4)2 = 3·2 +
(4+3)2 = +

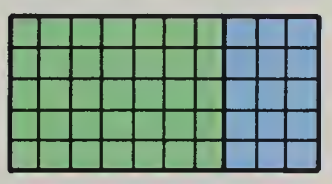


7 + 3
or 3 + 7

5·7
or 7·5

3·5
or 5·3



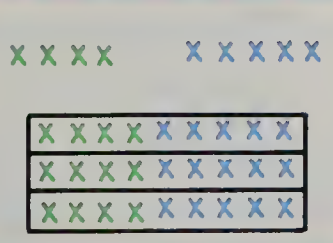


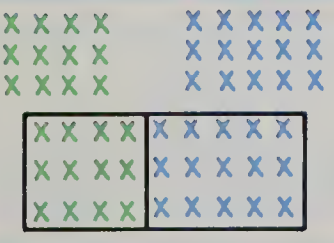
5(7+3)
or 5(3+7)
or (7+3)5
or (3+7)5

5·7 + 5·3
5·3 + 5·7
7·5 +
3·5 +

Conclusions:

5(7+3) = 5·7 +
5(3+7) = 5·3 +
(7+3)5 = 7·5 +
(3+7)5 = 3·5 +





Conclusions:

3(4+5) = 3·4 +
3(5+4) =
(4+5)3 = 4·3 +
(5+4)3 =

Because the pattern involved here is the basis for many time-saving shortcuts, it deserves further consideration.

Ben Harris once used arrows to focus attention on the way in which the pattern worked.

8(4+7) = 32 + ___ = ___

9(20+7) = 180 + ___ = ___

Ben called this his

"Do-It-to-Both" rule.

Ben pointed out that he could expand on this basic pattern in several ways.

7(2+5+3) = 14 + ___ + ___ = ___

6(100+70+8) = 600 + ___ + ___ = ___

Ben's "Do-It-to-All-Three" version.

And, of course, he has a "Do-It-to-All-Four" version.

7(8+3+7+2) = 56 + ___ + ___ + ___ = ___

Ben also had what he called his "vice versa" versions of the basic "Do-It-to-Both" rule.

(9+11)4 = 36 + 44 = ___

(70+4)7 = ___ + ___ = ___

(100+20+3)2 = ___ + ___ + ___ = ___

At first, Ben knew only the multiplication facts as far as $5 \cdot 5 = 25$, and he used his rule to help with more difficult problems.

3·9 = 3(5+4) = ___ + ___ = ___

4·12 = 4(5+5+2) = ___ + ___ + ___ = ___

When he learned the facts as far as $10 \cdot 10 = 100$, he kept right on using his rule.

8·34 = 8(30+4) = ___ + ___ = ___

7·98 = 7(+8) = ___ + ___ = ___

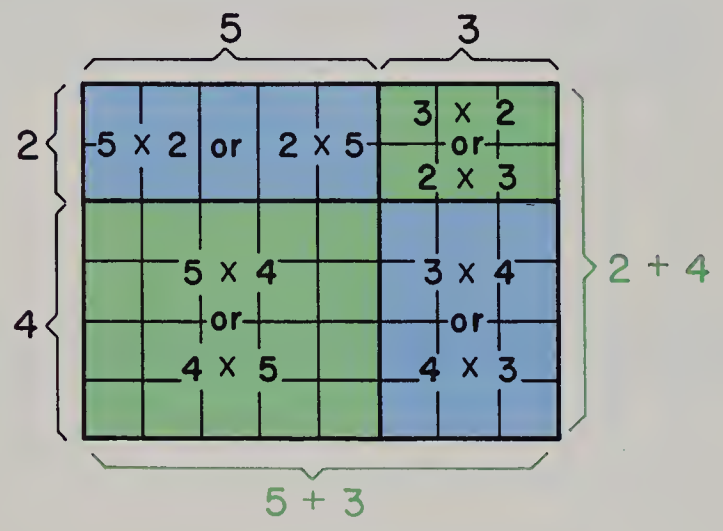
Another version of Ben's "Do-It-to-Both" rule he called "Double-Do-It-to-Both."

$(5+3)(2+4) = 10 + 20 + 6 + __ = ____$

$(8+5)(3+7) = 24 + __ + __ + __ = ____$

$(10+4)(10+8) = 100 + __ + __ + __ = ____$

This expansion of the "Do-It-to-Both" rule can be suggested in diagrams such as the following:



There are two ways in which we can account for all the blocks in the diagram:

A – $(5+3)(2+4) = __ \cdot __ = __$

B – $5\cdot2 + 5\cdot4 + 3\cdot2 + 3\cdot4 =$

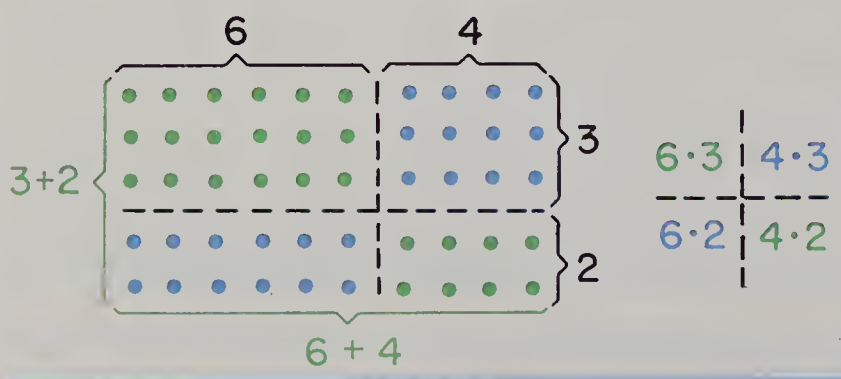
$10 + 20 + ____ + ____ = ____$

Conclusion:

$(5+3)(2+4) = 5\cdot2 + 5\cdot4 + 3\cdot2 + 3\cdot____$

This is the same conclusion Ben reached by his "Double-Do-It-to-Both" version.

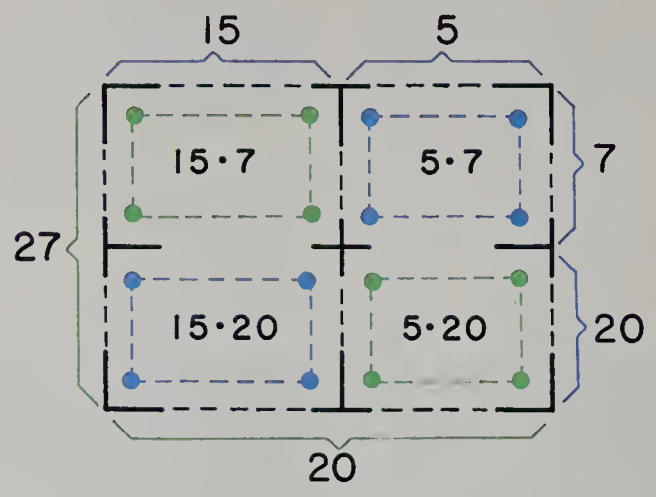
Let's try another example:



Conclusion:

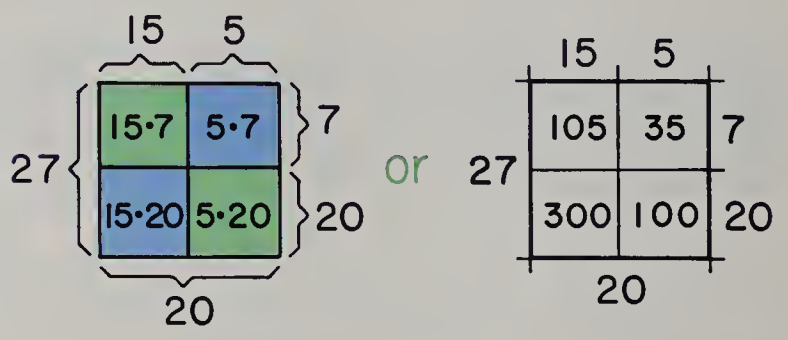
$(6+4)(3+2) = 6\cdot3 + 6\cdot2 + 4\cdot____ + ____$

Examples can involve such large numbers that diagrams would take up too much space if drawn in detail. Here, then, is a summary of another example:

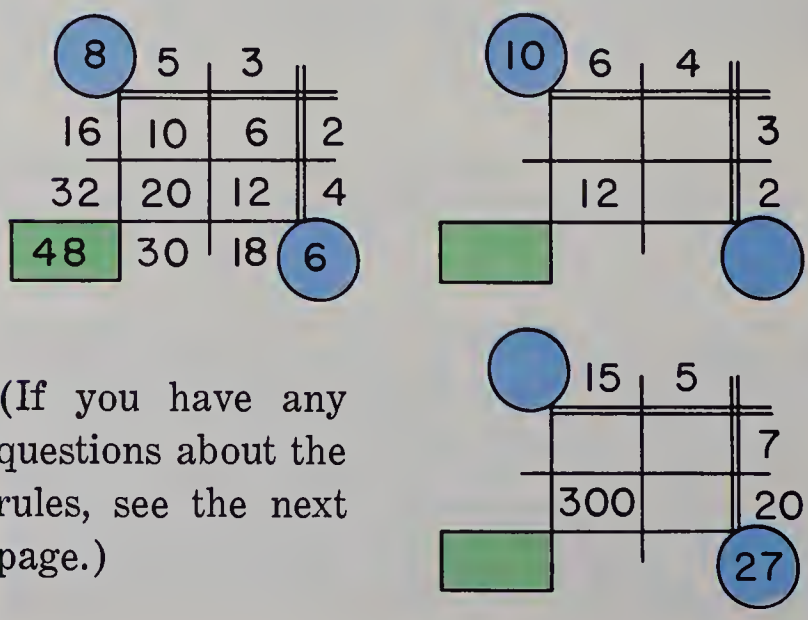


Broken green and blue lines above indicate that, if all the dots were drawn, there would be as many dots in that row or column as the number indicates.

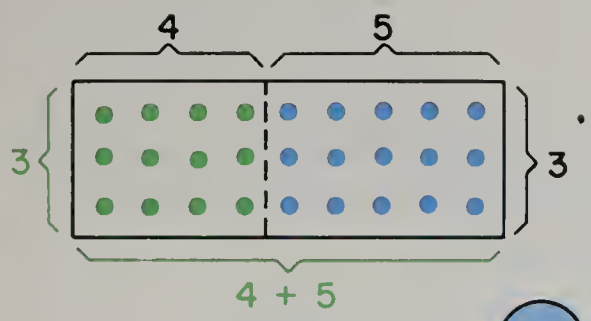
An even more condensed diagram might be:

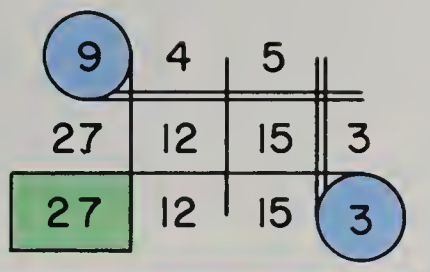


Ben, at this point, said that this suggested a version of the Cross Number Puzzle to keep a record of diagrams for his "Double-Do-It-to-Both" rule. He would record the three examples on the page in this way:



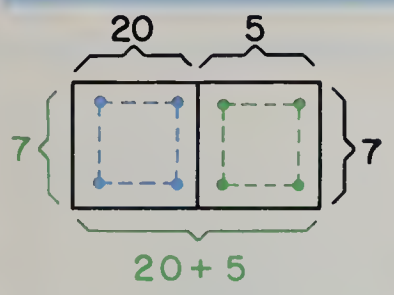
(If you have any questions about the rules, see the next page.)

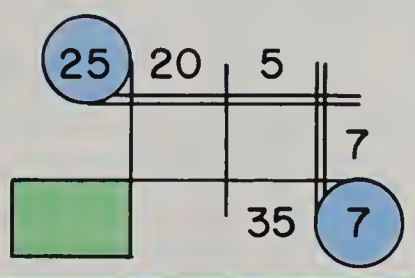




Conclusion:

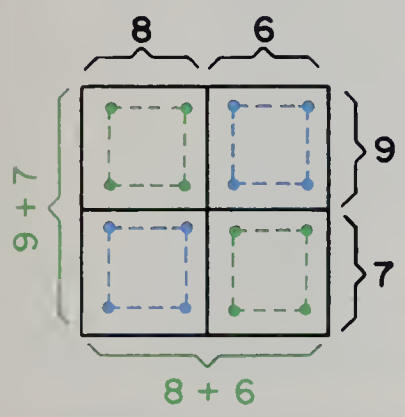
$3(4+5) = 3 \cdot 4 + 3 \cdot ___ = ____$

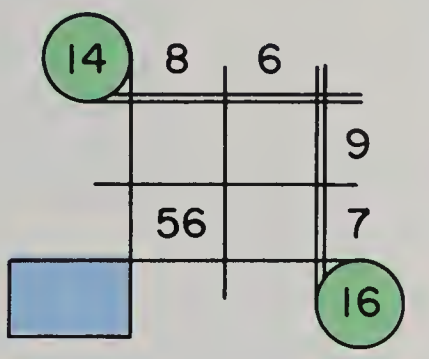




Conclusion:

$7(20+5) = 7 \cdot 20 + ___ = ____$

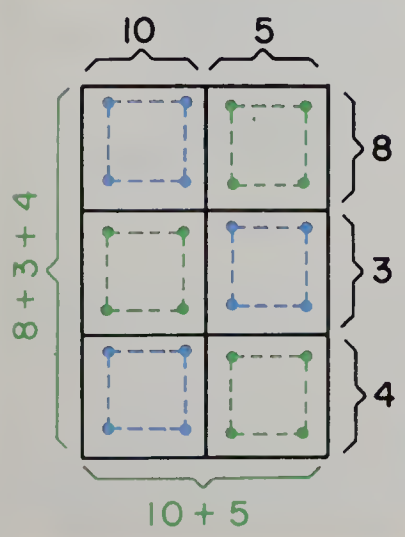


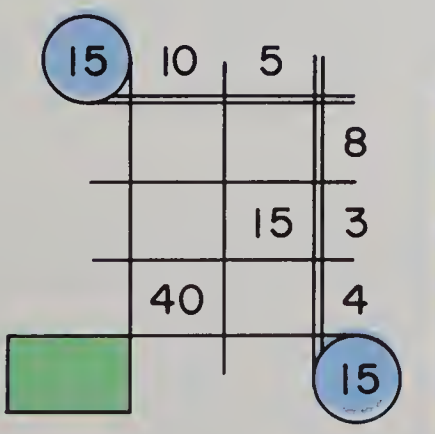


$14 \cdot 16 = ____$

Conclusion:

$(9+7)(8+6) = 9 \cdot 8 + 9 \cdot 6 + 7 \cdot ___ + ____$





$15 \cdot 15 = ____$

Conclusion:

$(10+5)(8+3+4) =$

$10 \cdot 8 + 10 \cdot 3 + 10 \cdot 4 + 5 \cdot 8 + 5 \cdot 3 + 5 \cdot 4$

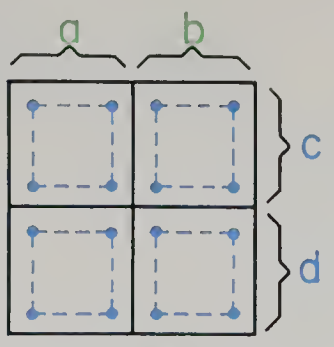
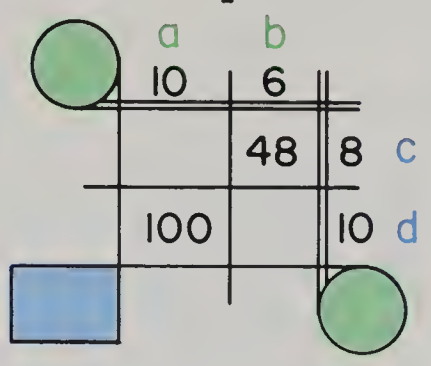


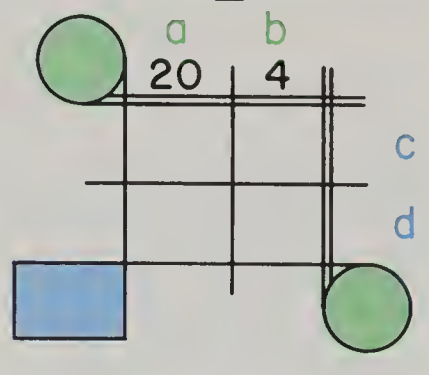
Chart and diagram indicate numbers to be used below.

	a	b	c	d
I	10	6	8	10
II	20	4	2	20
III	10	3	3	20
IV	10	7	2	10
V	40	1	2	30
VI	10	5	9	10

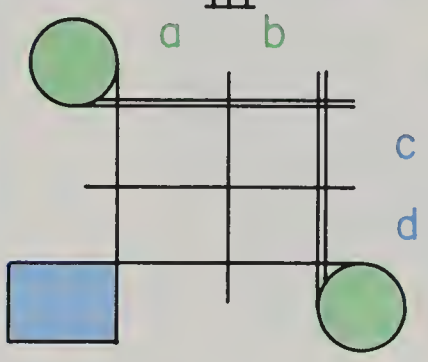
I



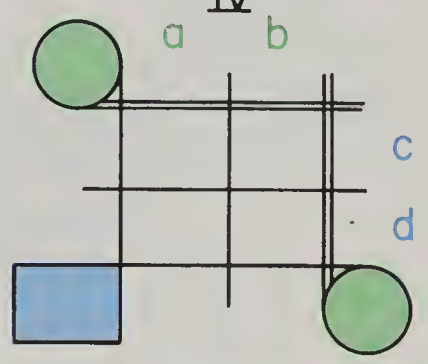
II



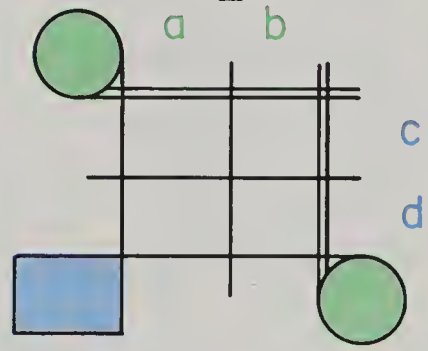
III



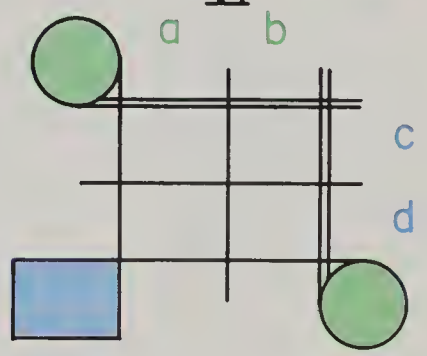
IV



V



VI



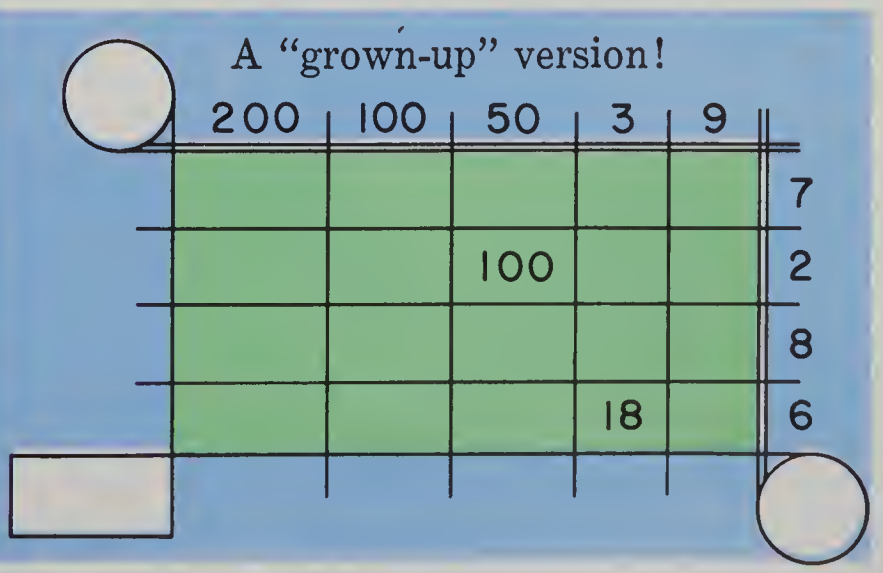
The examples above suggest the idea that:

I $16 \cdot 18 = ____$ II $24 \cdot 22 = ____$

III $___ \cdot ___ = ____$ IV $___ \cdot ___ = ____$

V $___ \cdot ___ = ____$ VI $___ \cdot ___ = ____$

A "grown-up" version!

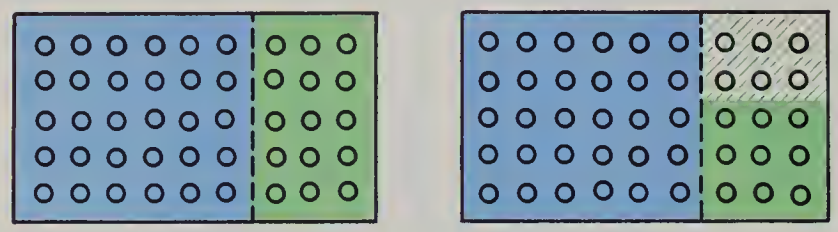


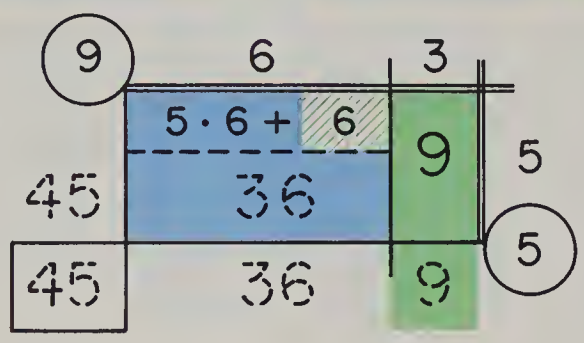
A Remembering Game

Marty, Ben Harris' friend, worked out a set of rules for a "remembering" version of the Cross Number Puzzle.

His first rule was: No number can be written in a space tinted green unless it is less than 10. The rest of the rules will become evident in a few examples.

On the right below is the way Marty thought about the diagram on the left.

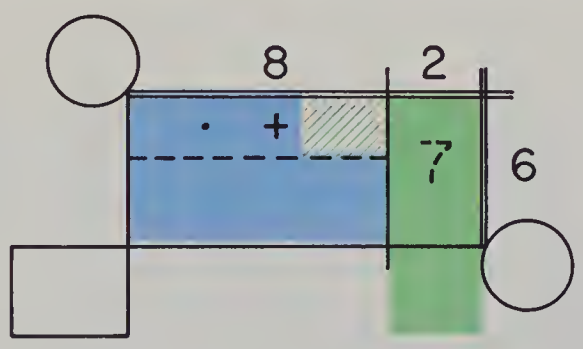




Marty would say: "3 · 5 or 5 · 3 equals 15 . . . but I can't write any number greater than 10 in the green space, so I'll write 9 and remember that I have 6 to account for later.

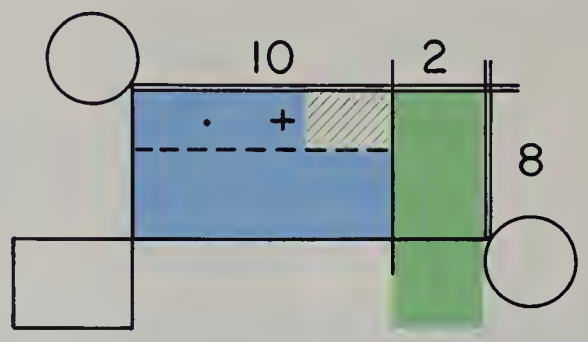
"Now, 5 · 6 = 30; and, adding the 6 that must be accounted for, since 30 + 6 = 36, I'll write 36 in the blue-tinted block . . . and proceed as with other Cross Number Puzzles."

Try an example without the diagram.



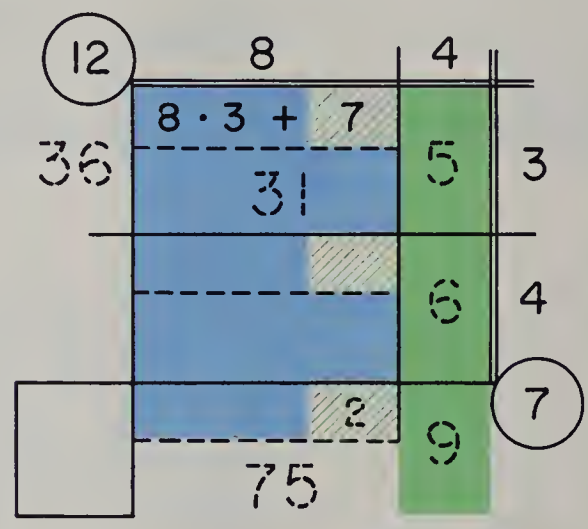
Since 6 · 2 or 2 · 6 equals 12, and 12 is not less than 10, suppose you decide to argue that 12 = 7 + 5, so you will write down 7 and remember that you must account later for the 5. Can you complete the example?

Try another completely on your own before the game gets more complicated.



Write any number you choose in the green-tinted space — provided it is less than 10.

Now, move on to a version that is something like two of the examples above — stacked one on top of the other.



You know that 3 · 4 = 12. Suppose you wrote 5 in the green block and decided to remember you must account for 7 more.

You know that 4 · 4 = 16. Suppose you decide to record 6 and remember that you must account later for 10.

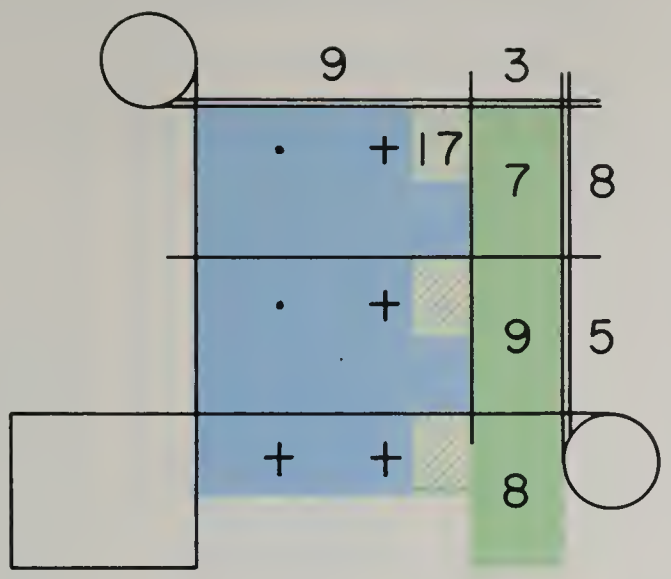
Now complete the work across these two rows: 36 = 31 + 5 and 48 = 42 + 6.

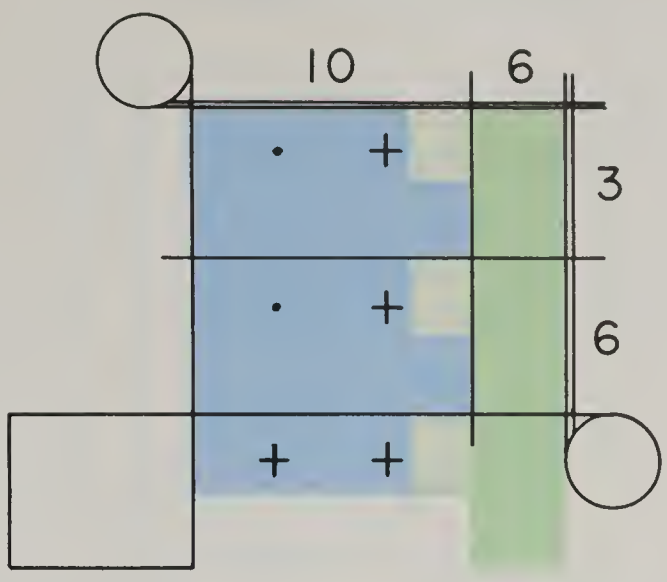
Consider next the green column. 5 + 6 = 11, but that is more than 10 and cannot be written in a green block.

So, you might record 9 and remember that 2 must be accounted for later.

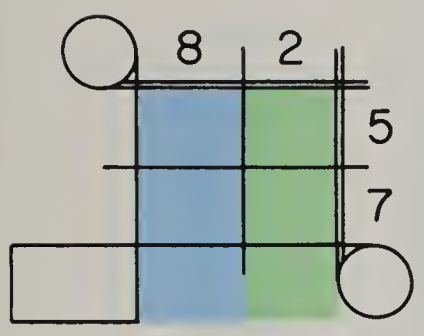
In the blue column, you know that 31 + 42 = 73, but you have 2 more to account for later, and 73 + 2 = 75.

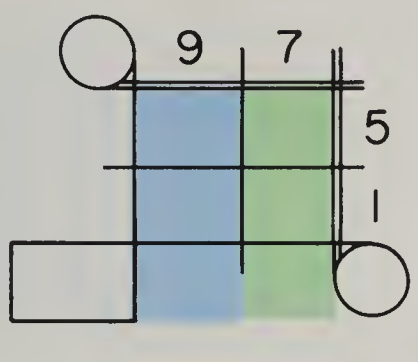
Add the bottom row and left-hand column, and the puzzle is completed.

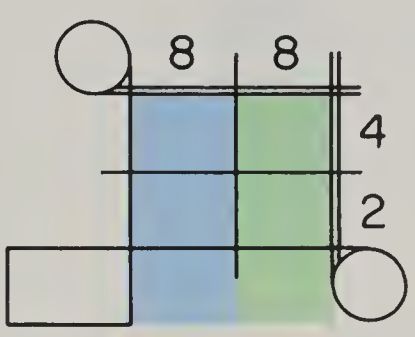


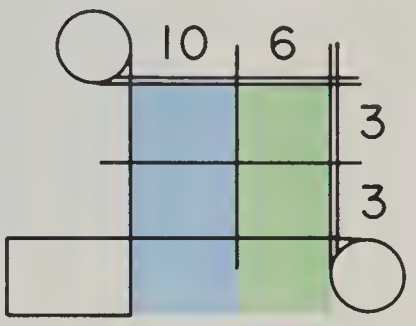


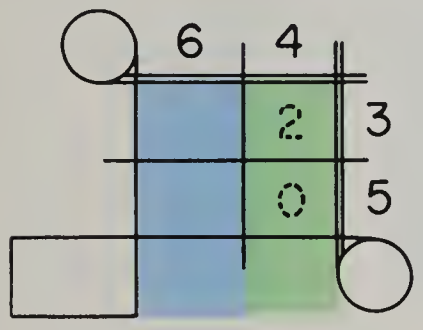
Marty wanted to save space and make his game more difficult. So he required that the players do their own "remembering" — in their heads or on another piece of paper.

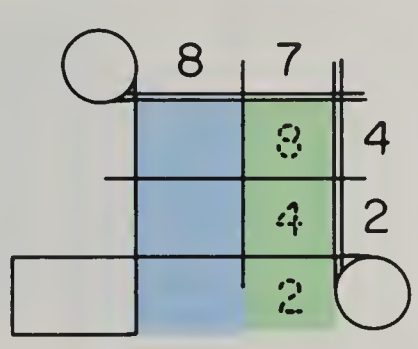


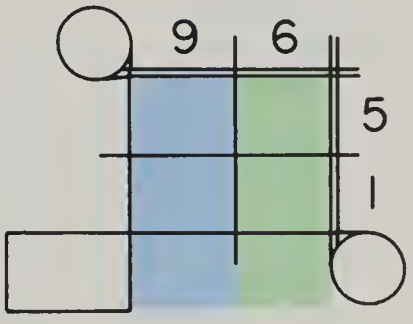


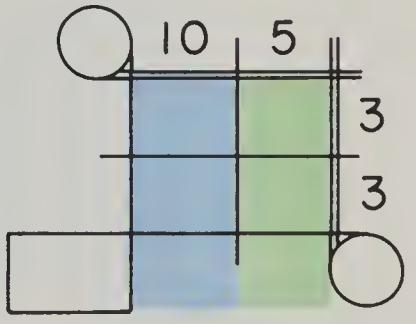


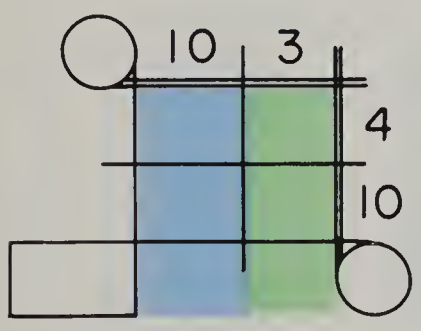


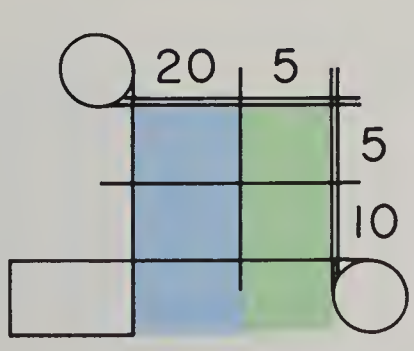


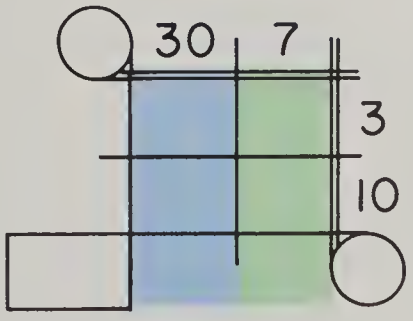


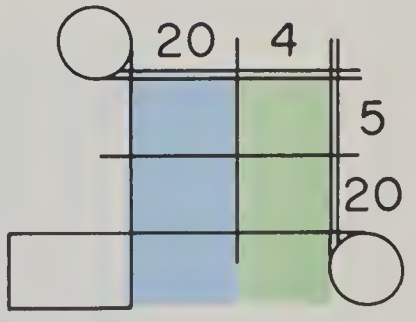












Another rule: You must record a number less than 10 (including 0) in green, but the numbers to be remembered and accounted for later must be multiples of 10 (including 0 · 10).

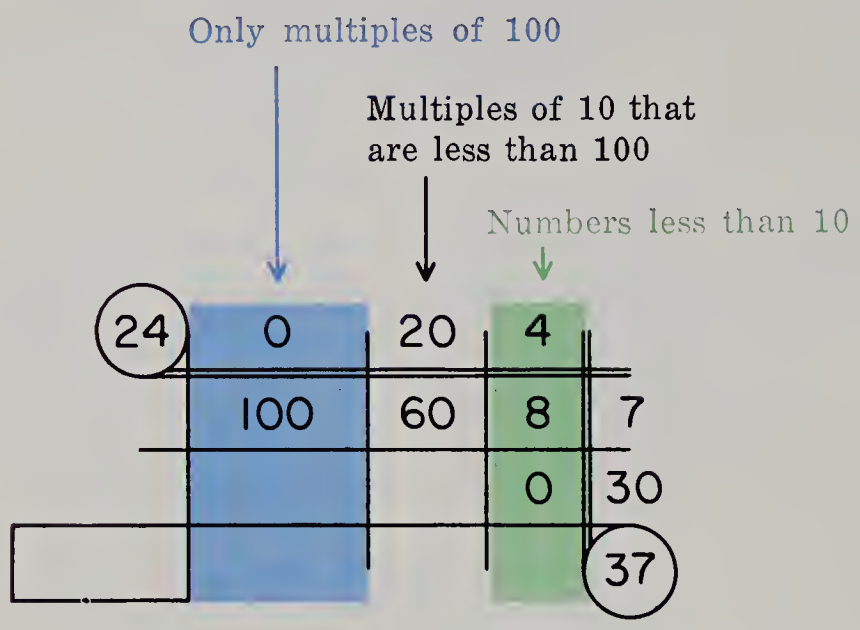
Carry out the following examples of multiplication:

10	15	13	25	37	24
x 8	x 6	x 14	x 15	x 13	x 25

Do you see any similarity with the Cross Number Puzzle above?

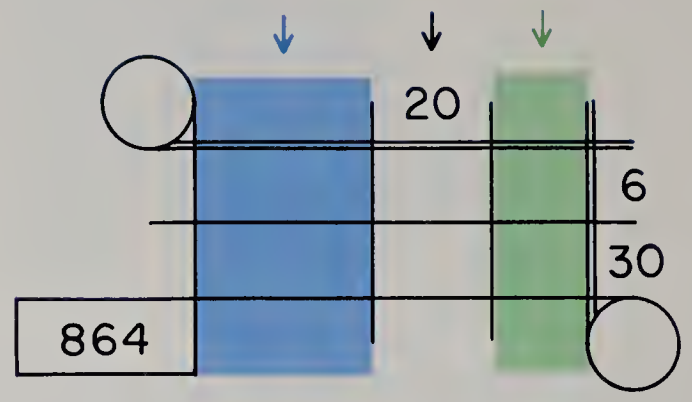
Marty's final set of rules:

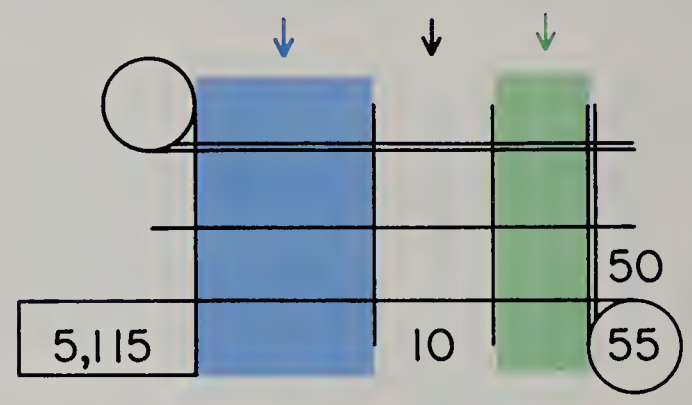
In the green column, only numbers less than 10 can be written; in the next column, only multiples of 10 that are less than 100 can be written; and, in the blue column, only multiples of 100 can be written.

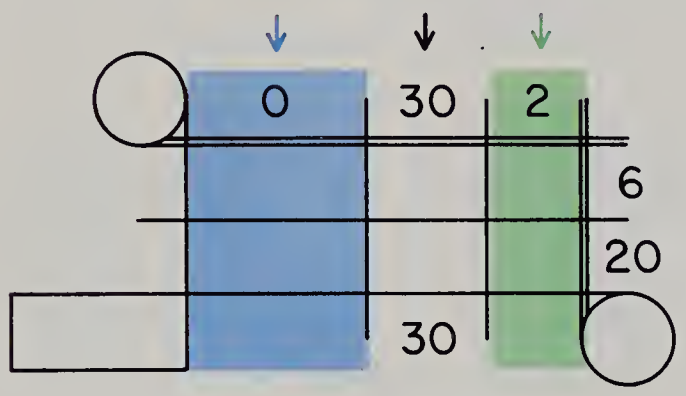


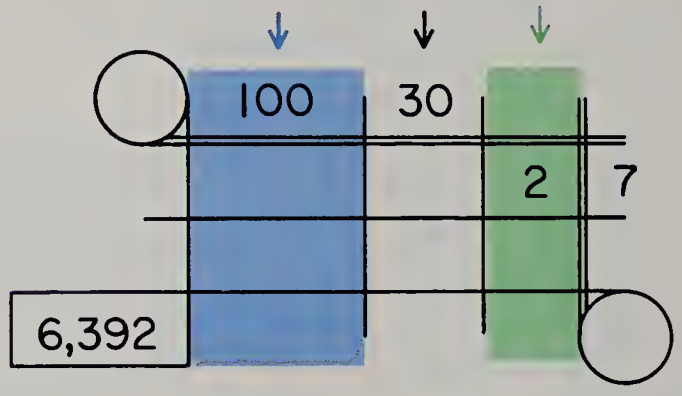
Other examples:

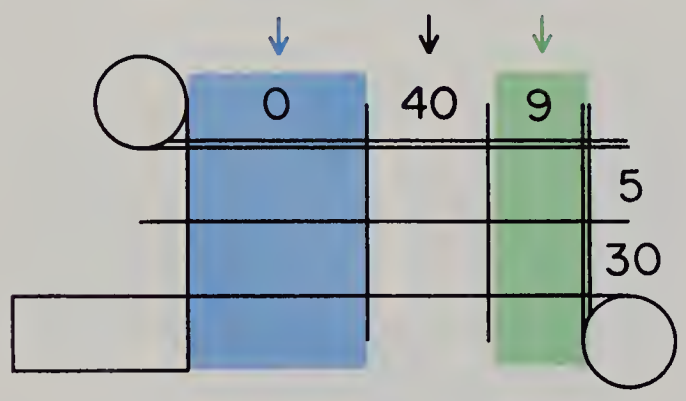
Below are some examples of this version of Marty's rules. Following these rules, there is only one way to reconstruct each example!

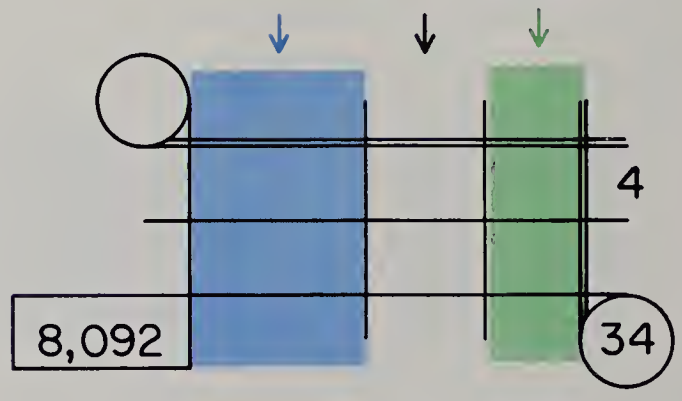


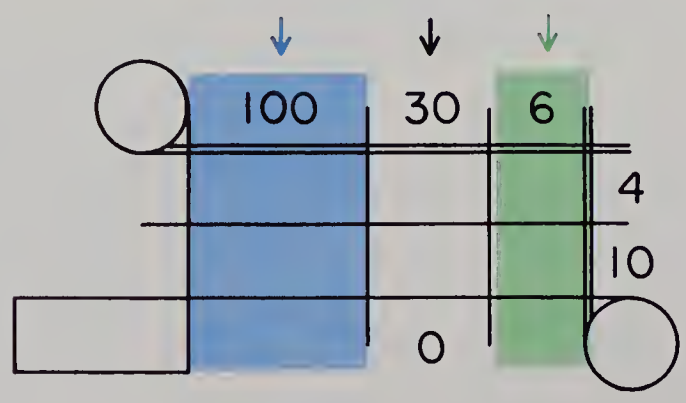


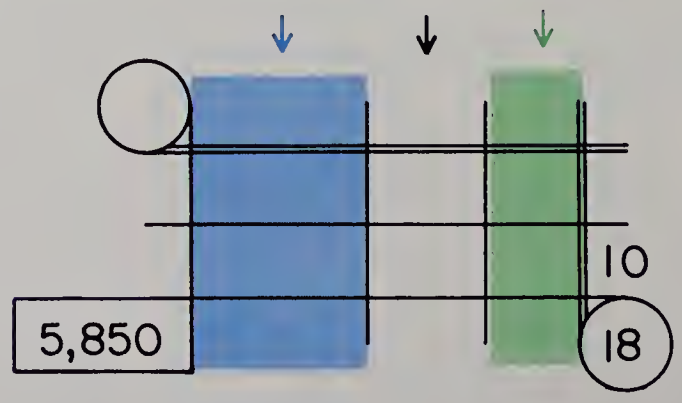












Complete the following:

49 × 35 =	32 × 26 =	24 × 37 =	136 × 14 =
36)864	55)5,115	136)6,392	
18)5,850	34)8,092		

Joan was inspired by Marty's games to invent a series of her own.

Joan's First Game

Rules: Fill in the blanks in each example so the sentences are true — but do not use multiples of 5.

3 (13 + ___) = 60	2 (___ + ___ + ___) = 28
7 (___ + ___) = 140	8 (___ + ___ + ___) = 136
9 (___ + ___) = 198	12 (___ + ___ + ___) = 144
15 (___ + ___) = 150	25 (___ + ___ + ___) = 725

Joan's Second Game

Rules: On the green line, use a multiple of 5, and on the black line do not use a multiple of 5. All sentences must be true.

3 (10 + ___) = 39	7 (___ + ___ + ___) = 140
8 (___ + ___) = 176	9 (___ + ___ + ___) = 315
10 (___ + ___) = 360	11 (___ + ___ + ___) = 275
15 (___ + ___) = 210	25 (___ + ___ + ___) = 500

Joan's Third Game

Rules: All numbers used to make each sentence true must all be odd or must all be even.

4 (12 + ___) = 104	7 (___ + ___ + ___) = 154
6 (___ + ___) = 216	8 (___ + ___ + ___) = 240
9 (___ + ___) = 306	12 (___ + ___ + ___) = 324
12 (___ + ___) = 216	35 (___ + ___ + ___) = 1,050

Joan's Fourth Game

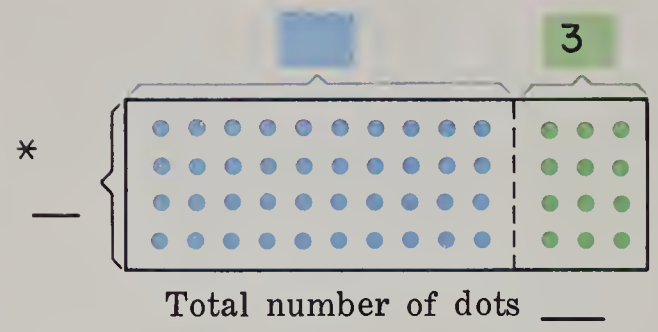
Rules: All numbers written above the green line must be multiples of 10; numbers written above the black line must be less than 10. All sentences must be true.

8 (___ + 6) = 208	5 (___ + ___ + ___) = 130
12 (___ + ___) = 300	4 (___ + ___ + ___) = 180
9 (___ + ___) = 495	7 (___ + ___ + ___) = 588
11 (___ + ___) = 792	13 (___ + ___ + ___) = 559
25 (___ + ___) = 375	15 (___ + ___ + ___) = 1,275
34 (___ + ___) = 340	18 (___ + ___ + ___) = 486
18 (___ + ___) = 324	16 (___ + ___ + ___) = 976
17 (___ + ___) = 510	24 (___ + ___ + ___) = 3,120

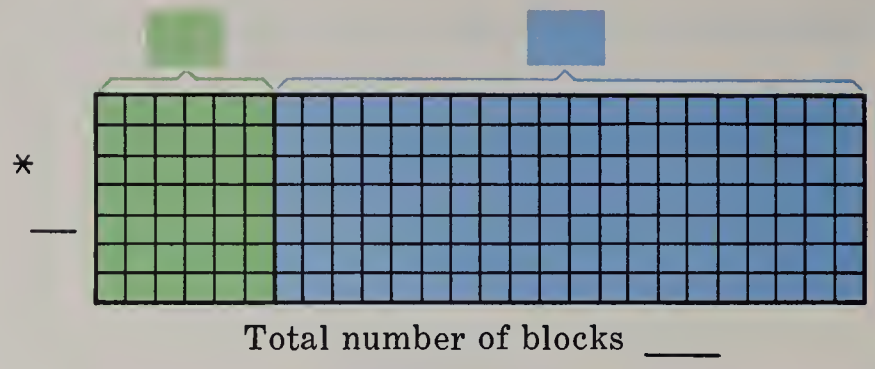
Joan's Fifth Game

Rules: Same as in fourth game, except that multiples of 10 less than 100 are written above the green lines, and the blue lines are reserved for multiples of 100.

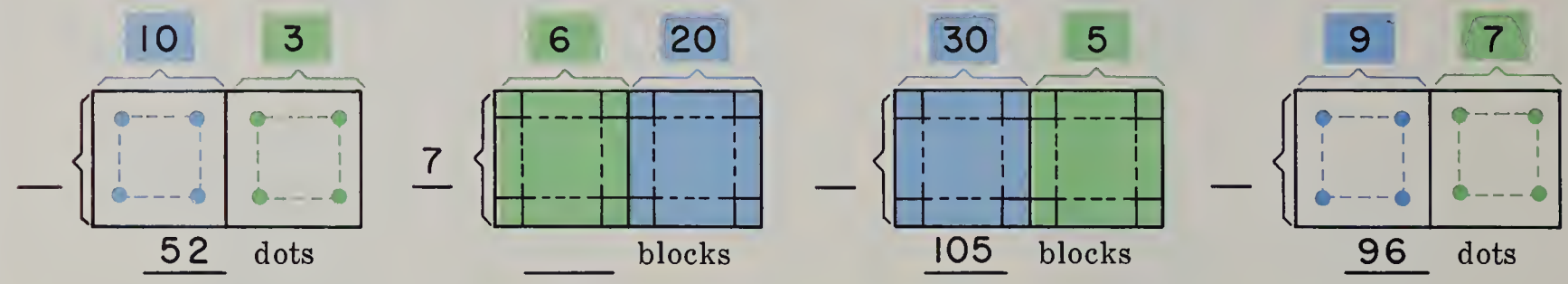
2 (___ + 40 + ___) = 486	
5 (___ + ___ + ___) = 590	
19 (___ + ___ + ___) = 4,180	
25 (___ + ___ + ___) = 12,700	
54 (___ + ___ + ___) = 20,304	



*(Number of rows, or dots in each column)

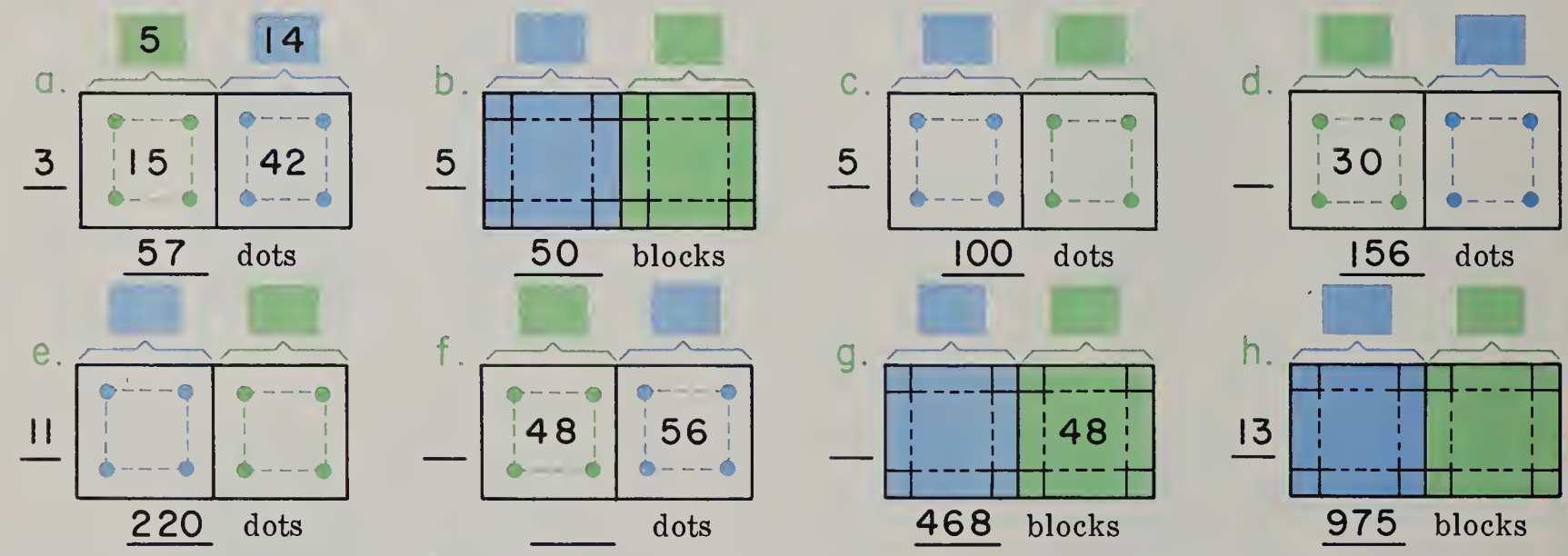


To save time and space, the sketches below are a kind of short-hand diagram of the same idea. Can you fill in the blanks?

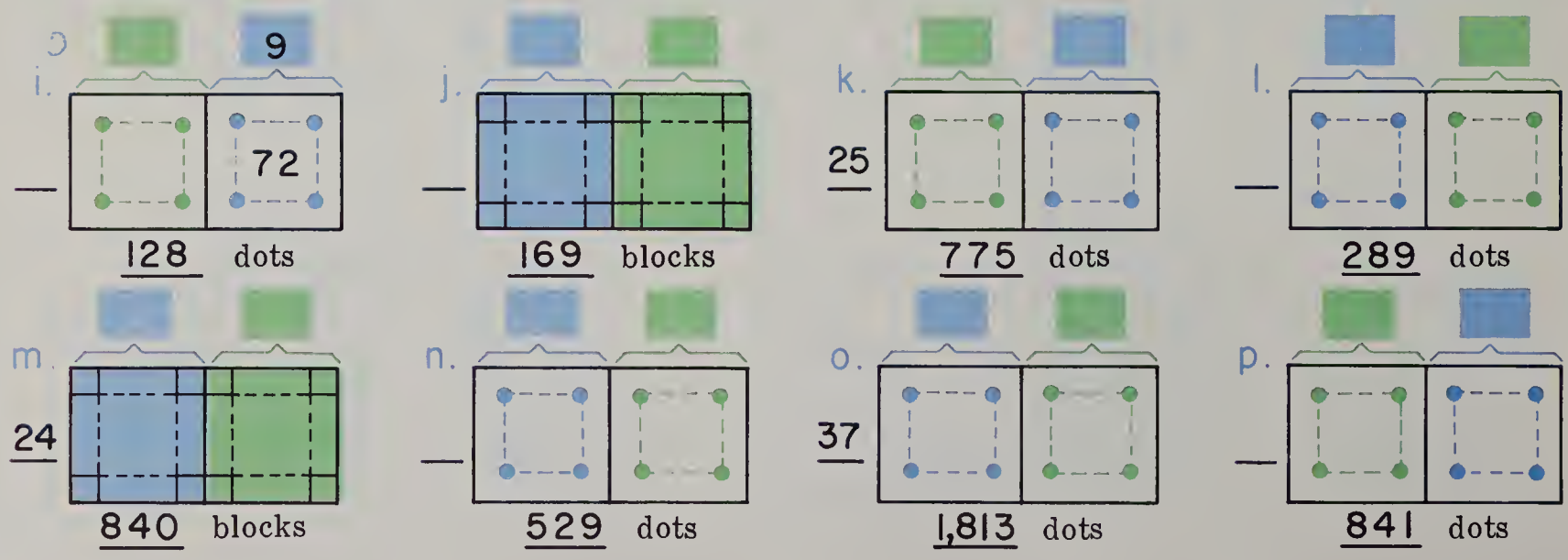


Betty's First Game

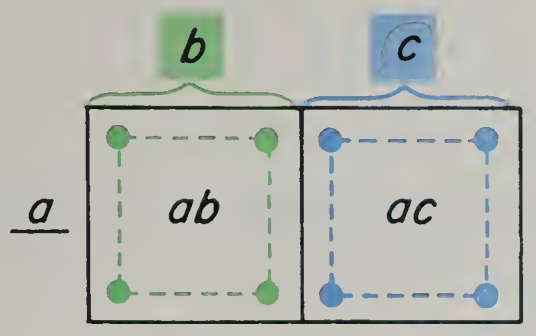
Betty used this kind of sketch for a game. Her rules were simple. Any number written in the blue box must be a multiple of 7, and numbers written in the green box must be less than 7. Also, the number of dots or blocks in each side should be indicated as in the first example.



Betty's Second Game. Number written in blue block to be a multiple of 9; in green, less than 9.



Betty got tired of drawing the same kind of diagram over and over again. So, she drew a single diagram and a chart. Can you find out the way in which she used them?



Total number of dots $a(b+c)$

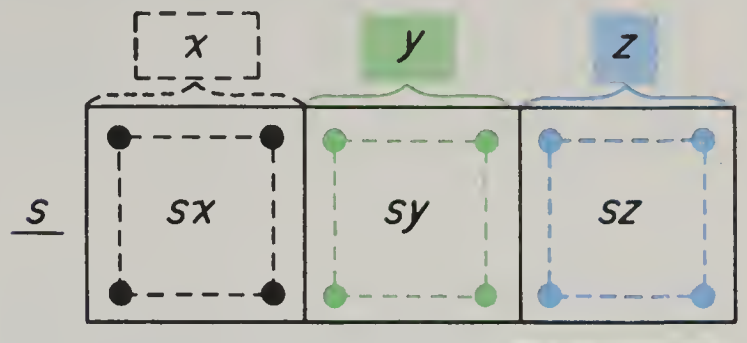
Betty's Third Game. Rules: Only multiples of 11 in green block (b) and numbers less than 11 in blue (c).

	a	b	c	ab	ac	$a(b+c)$
A		11	9			120
B	4					60
C	8					224
D	13					702
E			2		34	595
F	15			330		405

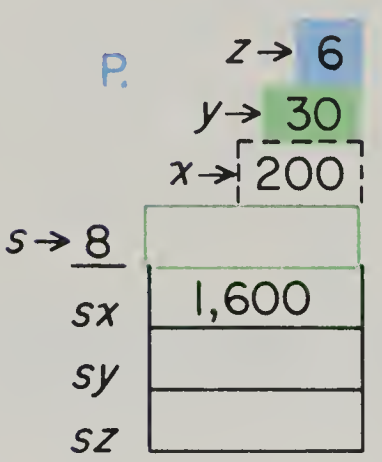
Betty's Fourth Game. Rules: Only multiples of 10 in green blocks (b) and numbers less than 10 in blue (c).

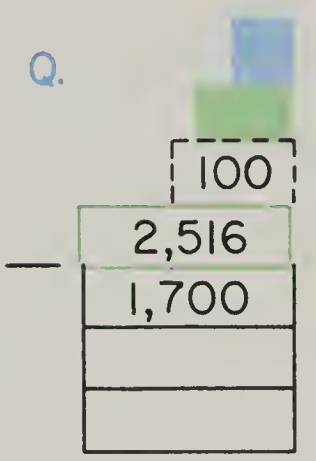
	a	b	c	ab	ac	$a(b+c)$
G	7					98
H	12					312
I	15		3	450		
J		20	8		136	
K	25					2,175
L	34					2,210
M	29		9	1,450		
N	38	30			266	
O	45					2,025

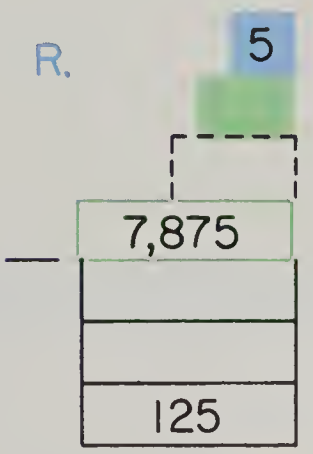
Next, Betty expanded her diagram. She added a rule to those of the fourth game: Only multiples of 100 can be written in the broken-line box. Betty also changed her chart.

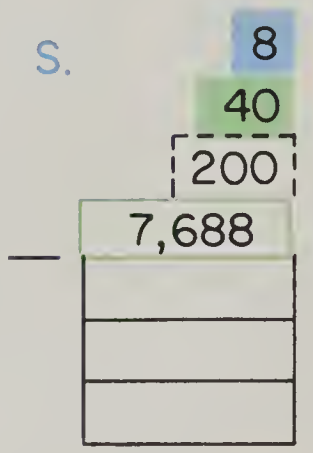


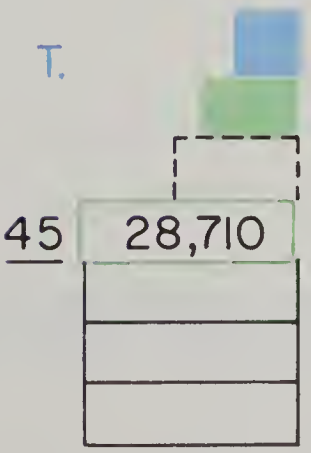
Total number of dots $s(x+y+z)$

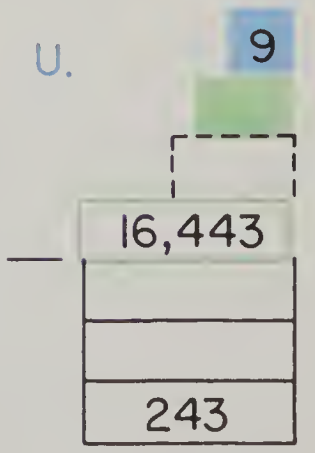




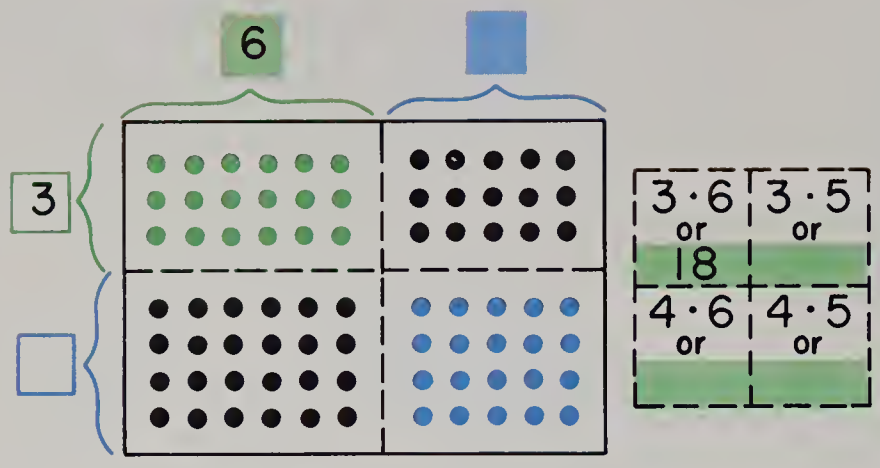






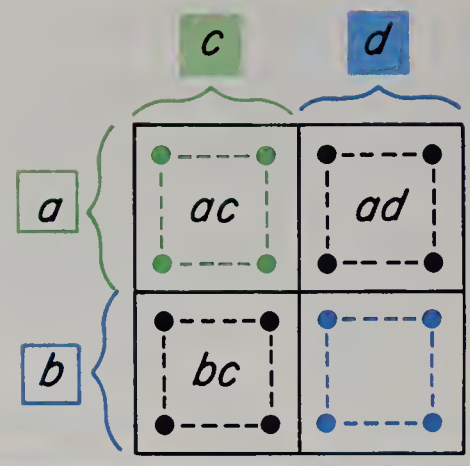


Betty's last game has a most familiar appearance to many people. Have you met this pattern before?



$18 + 15 + 24 + 20 =$ ____

A diagram and record of one example.



$ac + ad + bc + bd = (a+b)(c+d)$

A general diagram to be used for all examples in the chart below.

Study the example above of (3 + 4) (6 + 5) and its chart. A similar pattern can be seen in the general diagram at the right. Compare the letters with the chart below. The pattern in this diagram is used in all of the examples below. (No restrictions have been placed yet on the numbers that can be indicated in any of the boxes.)

Can you provide the missing information?

(Draw diagrams of any examples that trouble you.)

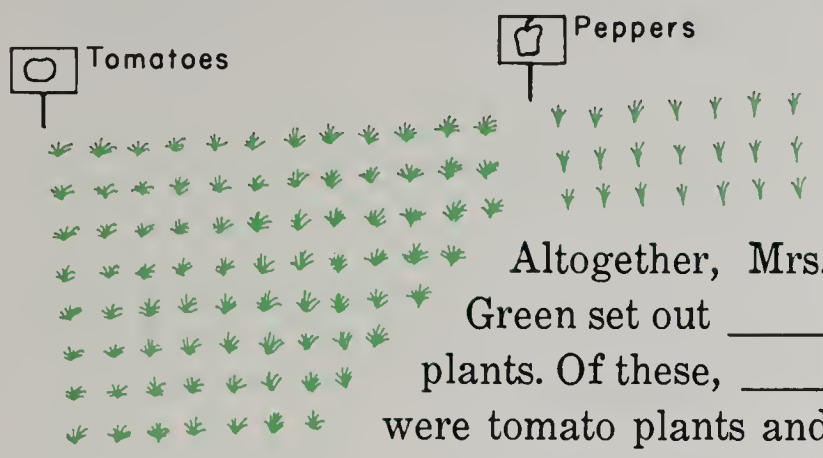
	a	b	c	d	ac	ad	a(c+d)	bc	bd	b(c+d)	(a+b)(d+c)
I	3	4	6	5			33			44	
II	6		8	2					8		
III			10	5		25		80			195
IV		5	3	6			360			45	
V	7	9							81		464
VI		4	30			140			28		888
VII				3		90	690		3	23	

A Game!

Rules: For the two green blocks (a and c), only multiples of 10 can be indicated; in the blue blocks (b and d), only numbers less than 10. Examples VI and VII above follow these rules.

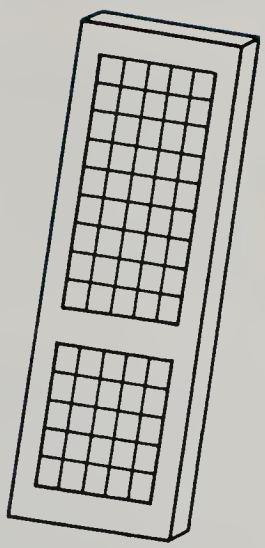
	a	b	c	d	ac	ad	a(c+d)	bc	bd	b(c+d)	(a+b)(c+d)
VIII	20	3	10	7							
IX	30	4		6				200			
X			20	5							675
XI		9		7		70	370	270			
XII	40	3							9		1,849
XIII			10	8			1,980			72	
XIV					1,500				35		2,035

In her garden, Mrs. Green set out 8 rows of plants. In each row, there were a dozen tomato plants and 7 pepper plants.



Altogether, Mrs. Green set out _____ plants. Of these, _____ were tomato plants and _____ were pepper plants. She set out ____ more tomato plants than pepper plants.

Mr. White wanted to put new panes of glass in a door. In the top section of the door there were 9 rows of 5 panes each. In the bottom section, 5 rows with 5 panes in each.



He needed a total of _____ panes; _____ for the top section and _____ for the bottom section.

Mrs. Swanson bought 8 cans of peas at 23 cents each, and 8 cans of corn at 27 cents each. For the cans of peas and corn she paid a total of $ _____ .

The "Do-It-to-Both" idea comes up often in similar situations:

No. of items	Price of each	No. of items	Price of each	Total Amount of Purchases

Make up your own stories (with or without sketches) that fit the "Do-It-to-Both" headlines below:

$5 \cdot 30 + 5 \cdot 70 = ___$

$25(28 + 42) = ___$

$(15 + 17)\,20 = ___$

$15 \cdot 25 + 18 \cdot 25 = ___$

$8(12 + 17 + 21) = ___$

The Grand Vizier invented the game of chess for King Shirham of India. After learning the rules, the King was fascinated by the game and wanted to reward the Grand Vizier (who was also a mathematician).

"Oh King," said the inventor, "give me two grains of wheat to put on one square of the chessboard, four for another square, eight for another, sixteen for another—and keep doubling the number of grains until we have accounted for the 64 squares on the board.

"That is a modest request," the King replied. "It is granted!" And he ordered his servants to bring in a bag of wheat.

The legend is told in many different ways. However, you can be sure that in no version is the King able to fulfill his promise.

You might guess that a chart something like the following was found in the Grand Vizier's notebook.

No. of squares	Number of grains for each square	Cumulative total
1st	2	2
2nd	4	6
3rd	8	14
4th	16	30
5th	32	62
6th	64	
7th	128	
8th		
9th		
10th		
11th		
12th		
13th		
14th		
15th		
16th		
17th		
18th		

A total of more than half a million grains of wheat — and only 18 of the 64 squares accounted for.

As he continued, the Grand Vizier saw no reason to keep the cumulative total. He had stopped adding the previous amount. (Can you see why?)

	Number of grains for each square
19th	524,288
20th	
21st	
22nd	
23rd	
24th	
25th	
26th	
27th	
28th	
29th	
30th	

Now the cumulative total has grown to _____

	Number of grains for the square indicated	Cumulative total of grains
64th	18,446,744,073,709,551,616	36,893,488,147,419,103,230

It has been estimated that there are about 5,000,000 grains of wheat in a bushel. The total wheat production of the world in a year is about 2,000,000,000 bushels.

Can you estimate the number of years it would take to fill the Grand Vizier's request if all wheat production were available?

We can only hope that the King and the Grand Vizier had a good laugh and then noticed some interesting facts about all those computations, such as:

1st square	2
2nd square	4
3rd square	8

2nd square	4
4th square	16
6th square	

2nd square	4
3rd square	
5th square	

3rd square	
4th square	
7th square	

2nd	4
2nd	4
4th	

9th	512
1st	
10th	

10th	
3rd	
13th	

12th	
2nd	
14th	

20th	1,048,576
3rd	
23rd	

6th	64
9th	512

5th	
5th	

	512
	4

	64
	64

	32
	256

	128
	4,096

	512
	512

2 5 7
4 x 32 =

32 x 64 =

32 x = 8,192

8 3 5
256 ÷ 8 =

1,024 ÷ = 64

16,384 ÷ 128 =

16 x = 256

32 x 64 =

8,192 ÷ 128 =

16 x 1,024 = _____

256 x 256 = _____

16,384 x 32 = _____

Because the Grand Vizier was also a skillful mathematician, there is hope that his interest turned to a study of the interesting results on the previous page.

Mathematicians have extended the results.

		Shorthand	is equal to
2 taken as a factor 0 times		2^0	1
2 taken as a factor 1 times	2	2^1	2
2 taken as a factor 2 times	2 · 2	2^2	4
2 taken as a factor 3 times	2 · 2 · 2	2^3	8
2 taken as a factor 4 times		2^4	16
2 taken as a factor 5 times		2^5	32
2 taken as a factor 6 times		2^6	
		2^7	
		2^8	
		2^9	
		2^{10}	

2^1 is read "2 to the 1st power"
2^2 is read "2 to the 2nd power" or "2 squared"
2^3 is read "2 to the 3rd power" or "2 cubed"
2^4 is read "2 to the 4th power"
2^5 is read "2 to the 5th power"
. . . . etc.

2^0 is read "2 to the zeroth power"— but what does it mean? It certainly fits the pattern in the "shorthand column."

In the column on the right, each number is half of the number below it. So, we hope that 2^0 equals $\frac{1}{2}$ of 2, or:

$2^0 = 1$

The following examples use 2^0 as another name for 1. Does it cause any trouble?

$2^2 \cdot 2^3 = 2^5$ because $4 \times 8 = 32$

$2^1 \cdot 2^6 = 2^7$ because $2 \times 64 =$ ____

$2^0 \cdot 2^3 = 2^3$ because $1 \times 8 =$ ____

$2^4 \cdot 2^3 =$ ___ because _____

$2^2 \cdot 2^4 =$ ___ because _____

$2^5 \cdot 2^0 =$ ___ because _____

$2^0 \cdot 2^0 =$ ___ because _____

$2^5 \div 2^3 = 2^2$ because $32 \div 8 = 4$

$2^6 \div 2^1 =$ ____ because _____

$2^5 \div$ ____ $= 2^1$ because _____

$2^6 \div 2^0 =$ ____ because _____

___ $\div 2^4$ 2^2 because _____

$2^8 \cdot 2^5 = 2^{13} = 8{,}192$

$2^{17} \cdot 2^3 =$ ____ = _____

$2^2 \cdot 8 =$ ____ = _____

$256 \cdot 64 =$ ____ = _____

Of course, the Grand Vizier, being a mathematician, would wonder if other tables could be built that would have a similar pattern.

Suppose we "triple" each time instead of "doubling."

3 as a factor 0 times		3^0	1
3 as a factor 1 times	3	3^1	3
3 as a factor 2 times	3·3	3^2	9
3 as a factor 3 times	3·3·3	3^3	
3 as a factor 4 times	3·3·3·3	3^4	
3 as a factor 5 times			
3 as a factor 6 times			

3^5	243
3^3	
3^8	

3^6	729
	81

	27
	2,187

	81
	177,147

	729
	729

	243
	1,594,323

3^3	27
3^2	9
3^5	

3^2	9
3^1	
3^3	

3^4	
3^0	
3^4	81

	81
	9

	27
	729

	81
	243

Let's extend the table

3^7	2,187
3^8	
3^{13}	

5 — 5 — 10

243 × 243 = ☐

2,187 × ☐ = 2,187

19,683 ÷ 243 = ☐

6,561 = 531,441 ÷ ☐

1,594,323 = 729 × ☐

☐ × 27 = 177,147

☐ ÷ 81 = 27

$$243 \overline{)\,531{,}441} \quad 2{,}187$$

$$\begin{array}{r} 729 \\ \times\ 243 \\ \hline \end{array}$$

	5^0	1
5	5^1	5
5·5	5^2	25
5·5·5	5^3	
	5^4	

	7^0	1
7	7^1	7
7·7	7^2	49
7·7·7	7^3	
	7^4	

	10^0	1
10	10^1	10
10·10	10^2	100

In the examples below, use the table as a shortcut if you want to.

5^2 25 × 5^3 125 = 5^5

□ 15,625 ÷ 5^4 = □

□ 390,625 = □ × 3,125

□ = 5^7 ÷ 5^3

□ 343 × 7^3 = □

7^4 ÷ □ = 7^1

□ ÷ 49 = 16,807

□ 16,807 × □ = 7^5

□ = 10,000 ÷ 10

□ 1,000,000 ÷ 10^2 = □

10^0 × 10^3 = □

10^2 × □ = 10^6

10^5 ÷ 10^3 = □

Astronomers have watched stars that are

$$2,500,000,000,000,000$$

miles distant from earth. But writing all those zeros is a waste of time. For 2,500,000,000,000,000, scientists write

$$25 \cdot 10^{14} \text{ or } 2.5 \cdot 10^{15}$$

This is a great time saver. Work through the next group of examples. You can see the pattern of their system. It depends on an extension and further use of the table at the bottom of the opposite page.

$1{,}000 = 10 \cdot 10 \cdot 10 = 10^3$

$7{,}000 = 7 \cdot 10 \cdot 10 \cdot 10 = 7 \cdot 10^3$

$21{,}000 = 21 \cdot 10 \cdot 10 \cdot 10 = 21 \cdot 10^3$

$10{,}000 = 10 \cdot 10 \cdot 10 \cdot 10 =$ _____

$80{,}000 = 8 \cdot 10$_____ $= 8 \cdot$ ____

$620{,}000 = 62 \cdot$_____ $=$ _____

$100{,}000 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 =$ _____

$50 \cdot 1{,}000 = 5 \cdot 10^1 \cdot 10^3 = 5 \cdot 10^4$

$100 \cdot 1{,}000 = 10^2 \cdot 10^3 =$ _____

$400 \cdot 1{,}000 = 4 \cdot 10^2 \cdot 10^3 = 4 \cdot$ ____

$1{,}200 \cdot 1{,}000 = 12 \cdot 10^2 \cdot$___ $= 12 \cdot$ ____

$3 \cdot 10^2 \cdot 5 \cdot 10^4 = 3 \cdot 5 \cdot 10^2 \cdot 10^4 = 15 \cdot$ ___

$7 \cdot 10^4 \cdot 8 \cdot 10^7 = 7 \cdot 8 \cdot$_____ $=$ _____

$3 \cdot 10^4 \cdot 7{,}000 = 3 \cdot 7 \cdot$_____ $=$ _____

$85{,}000 \cdot 10{,}000 = 85 \cdot$_____ $=$_____

$3{,}000 \cdot 800 = 3 \cdot 8 \cdot 10^3 \cdot 10^2 =$ _____

$900 \cdot 110{,}000 =$ _____ $=$ _____

$10^2 \cdot 10^5 \cdot 10^4 =$ _____

Astronomers (and other scientists, too) need shorthand ways to write large numbers. Here are a few examples:

Our earth is about 93,000,000 miles from the sun, or

$93 \cdot 10^6$ miles from the sun.

The furthest planet from the sun, Pluto, is about 40 times farther away.

$$40 \cdot 93 \cdot 10^6 = 4 \cdot 93 \cdot 10^7 = 372 \cdot 10^7$$

Saturn is only about 10 times as far from the sun as our earth is — or $10 \times 93{,}000{,}000$ miles.

$10 \cdot 93 \cdot 10^6 =$ _____

Before they used radio telescopes, astronomers could see only about 30,000,000,000 stars, or

$3 \cdot$____ stars.

One of the largest they could see has a diameter of 2,400,000,000 miles, or

$24 \cdot$____ miles.

Our moon varies its distance from the earth from between 221,000 to 253,000 miles. When it is 250,000 miles away, we write this as:

$25 \cdot$____ miles away.

Uranus, another planet, has four moons. Uranus is 1,800,000,000 miles from our sun, or

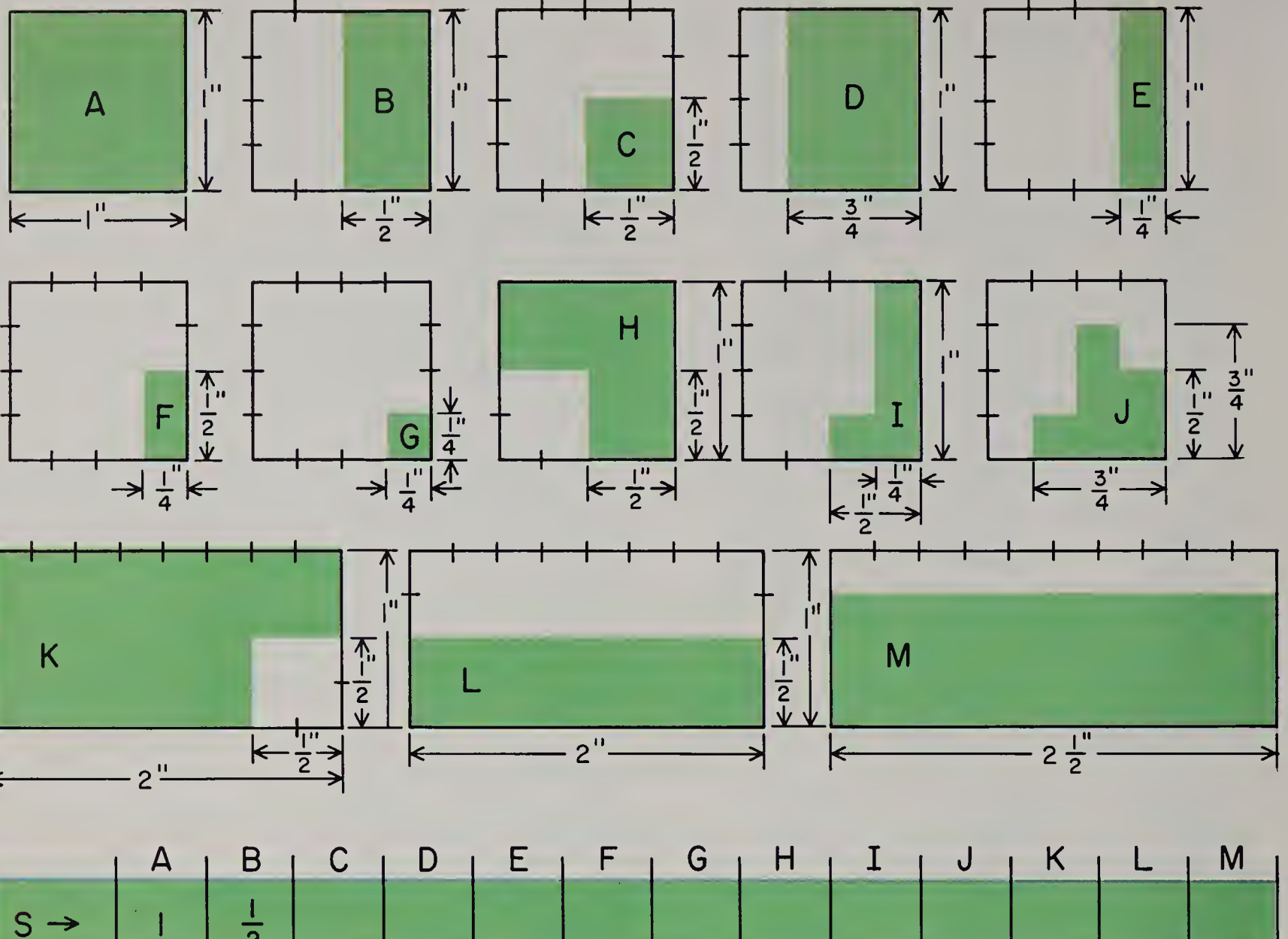
_____ miles from our sun.

Here is another scientific problem that requires a shorthand for writing large numbers:

How many atoms are there in the Universe ? — Approximately

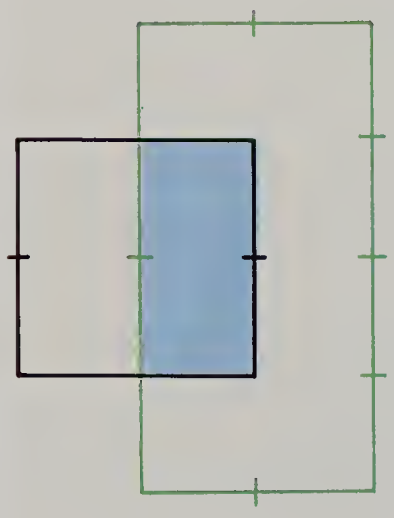
300,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000 — or

$3 \cdot 10^{74}$!



	A	B	C	D	E	F	G	H	I	J	K	L	M
S →	1	1/2											
U →	0												

S → number of square inches in shaded area; U → number of square inches in unshaded area.



What is the size, in square inches, of the area? . . .

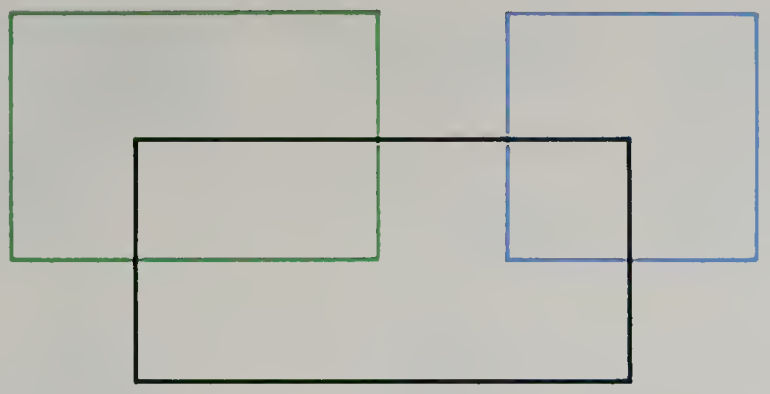
. . . inside the black line? _____
. . . inside the green line? _____
. . . inside the black, but not inside the green lines? _____
. . . inside the green, but not inside the black lines? _____
. . . inside both black and green lines? _____ (blue tint)
. . . inside either black or green lines? _____ (entire sketch)
. . . inside either black or green lines, but not in black and green? _____ (all except blue)

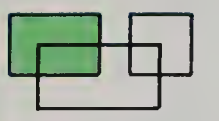
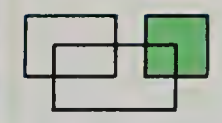
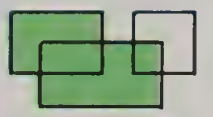
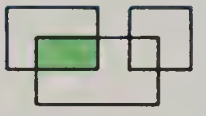
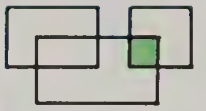
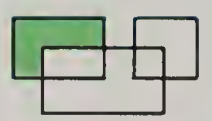
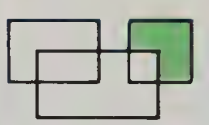
On the next page, you will need to remember this particular way we are using the words "and" and "or."

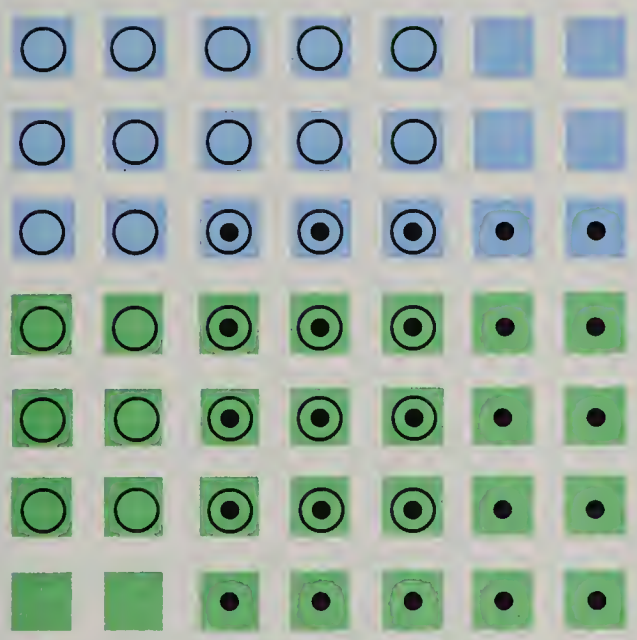
We will use "and" to express the idea of "in both" or in "all three" at the same time.

We will use "or" to express the idea of "in either" of two or "in any" of all three or more. Keep your eye on those words.

Use a ruler to determine the size of the rectangles.

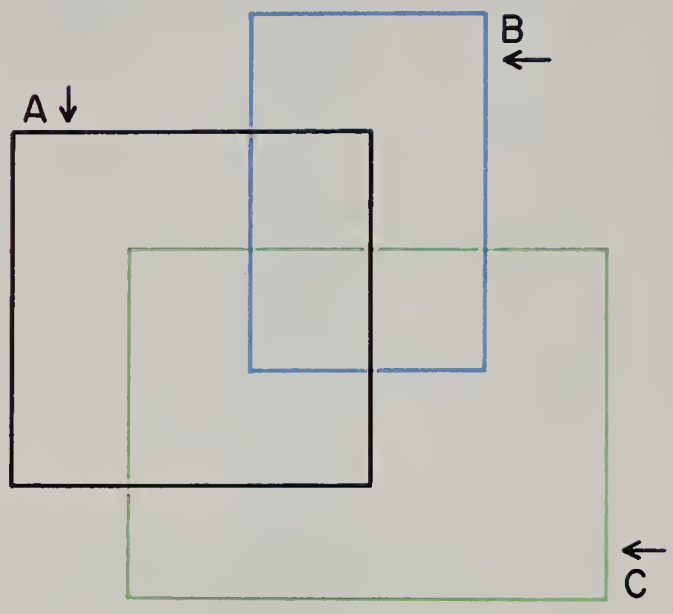


How many Square Inches . . .	Hints
(a) inside the green line?	
(b) inside the black line?	
(c) inside the blue line?	
(d) in either green or black?	
(e) in either black or blue?	
(f) in both green and black?	
(g) in both black and blue?	
(h) in green but not in black?	
(i) in blue but not in black?	
(j) in black but not in green or blue?	

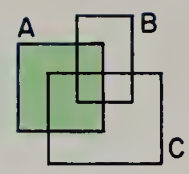


How many . . .	No.
blocks? (green or blue)	
green blocks?	
blue blocks?	
circles? (with or without dots)	
dots? (with or without circles)	
blocks without dots or circles?	
blue blocks with dots?	
blue blocks with circles?	
blue blocks with dots or circles?	
blue blocks with dots and circles?	
green blocks with dots?	
green blocks with circles?	
green blocks with dots or circles?	
green blocks with dots and circles?	
blocks (green or blue) with dots or circles?	
blocks with dots and circles?	
blue blocks without dots?	
green blocks without circles?	
blue blocks without circles?	
green blocks without dots and circles?	

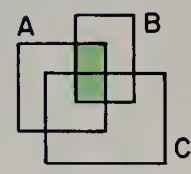
With a ruler, measure the lines in the diagram before considering the questions below.



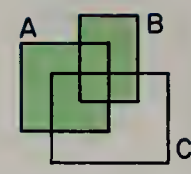
Hints and definitions



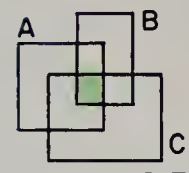
in A



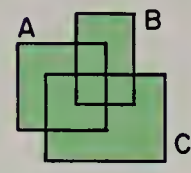
in A and B



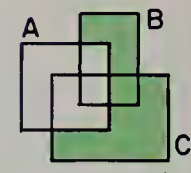
in A or B



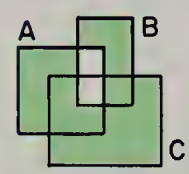
in A and B and C



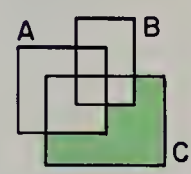
in A or B or C



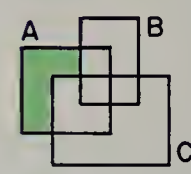
not in A



not in A and B



not in A or B

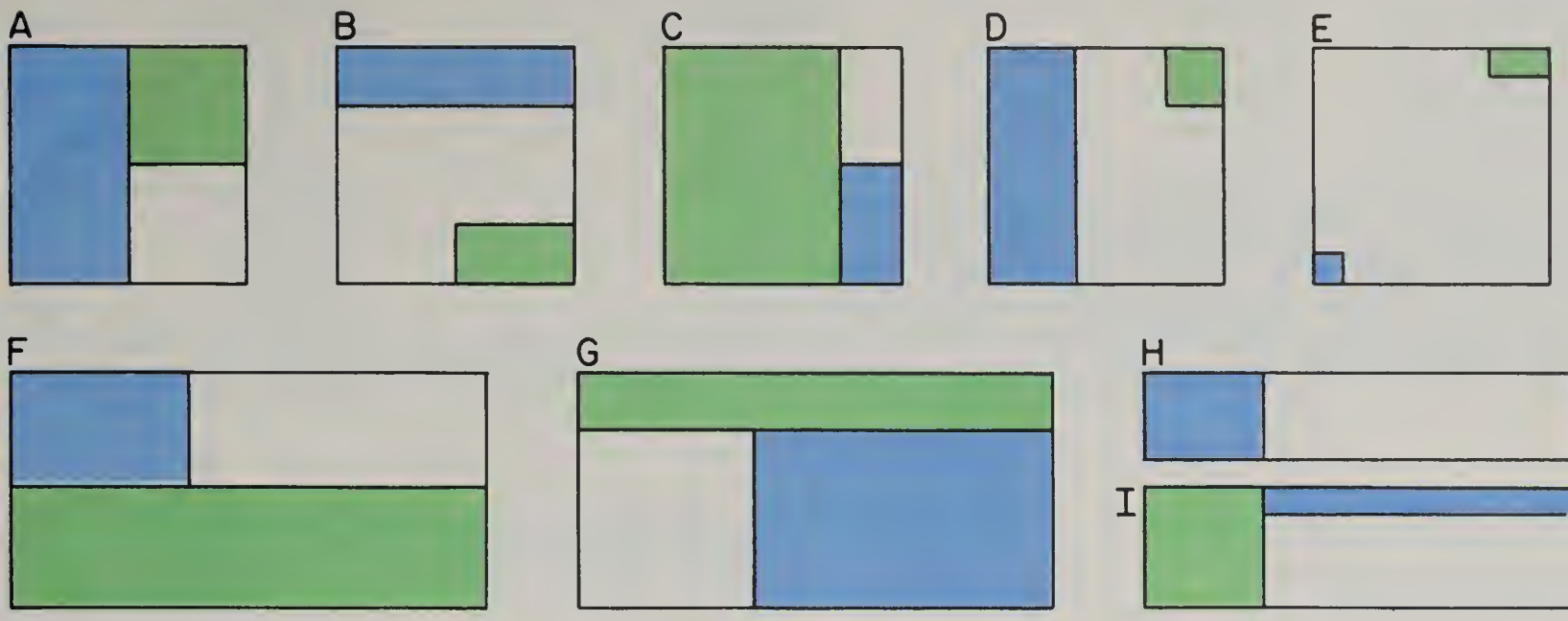


in A but not in B or C

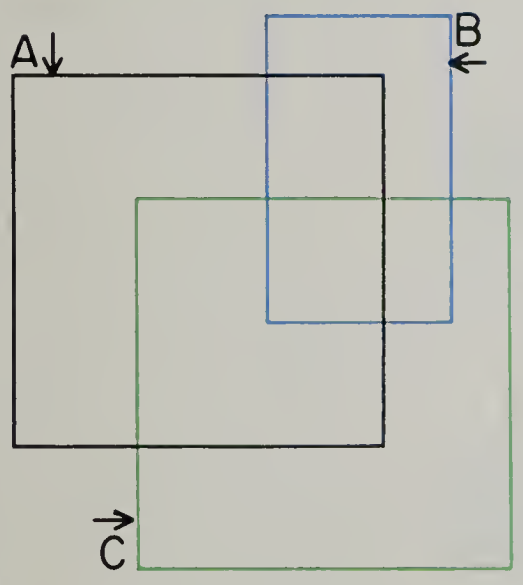
How many Square Inches . . .			Sq. In.
I	in A?	(see Hints)	
II	in B?		
III	in C		
IV	in A and B	(Hints)	
V	in A and C		
VI	in B and C		
VII	in A or B	(Hints)	
VIII	in A or C		
IX	in B or C		
X	in A and B and C	(Hints)	
XI	in A or B or C	(Hints)	
XII	not in A	(Hints)	
XIII	not in B		
XIV	not in C		
XV	not in A and B	(Hints)	
XVI	not in A and C		

			Sq. In.
XVII	not in B and C?		
XVIII	not in A or B?	(Hints)	
XIX	not in A or C		
XX	not in B or C		
XXI	in A but not in B		
XXII	in A but not in C		
XXIII	in B but not in A		
XXIV	in B but not in C		
XXV	in C but not in A		
XXVI	in C but not in B		
XXVII	in A but not in B or C	(Hints)	
XXVIII	in B but not in A or C		
XXIX	in C but not in A or B		
XXX	in A or B but not in C		
XXXI	not in A and B and C		
XXXII	not in A or B or C		

Use a ruler to determine the size of the areas in the following diagrams:



How many square inches in . . .	A	B	C	D	E	F	G	H	I
the area shaded blue?									
the unshaded area?									
the area shaded green?									

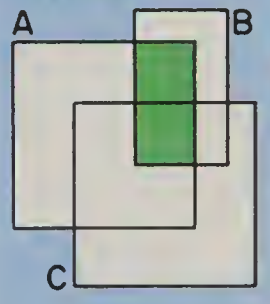


How many square inches in . . .

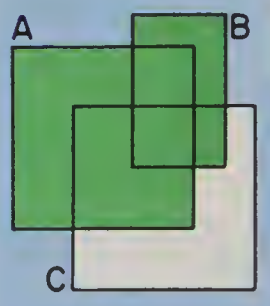
A?	
B?	
C?	
A and B	
B and C	
A and C	

A or B	
B or C	
C or A	
A but not in B	
A but not in C	
B but not in A	
B but not in C	
C but not in A	
C but not in B	
A but not in B or C	
B but not in A or C	
C but not in A or B	
A and B and C	
A or B or C	

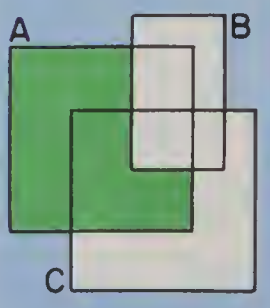
Hints



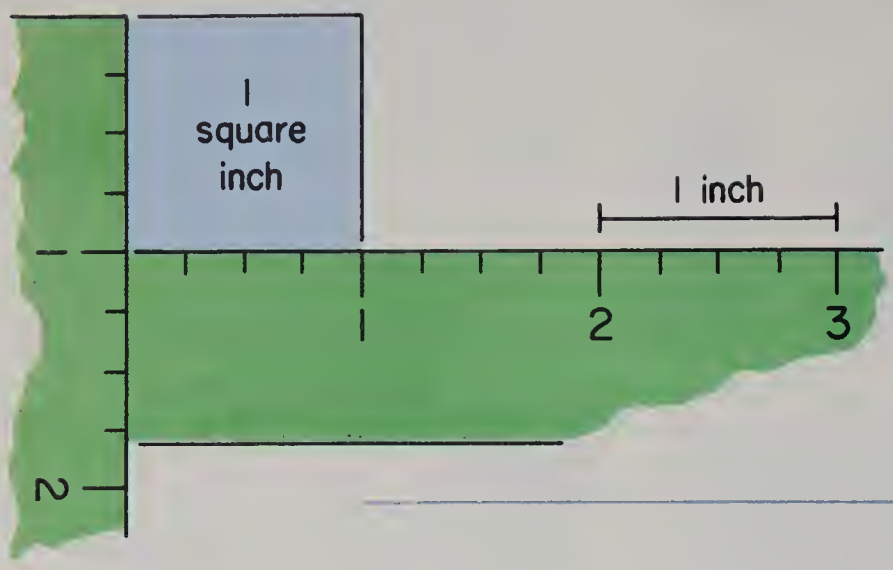
in A and B



in A or B



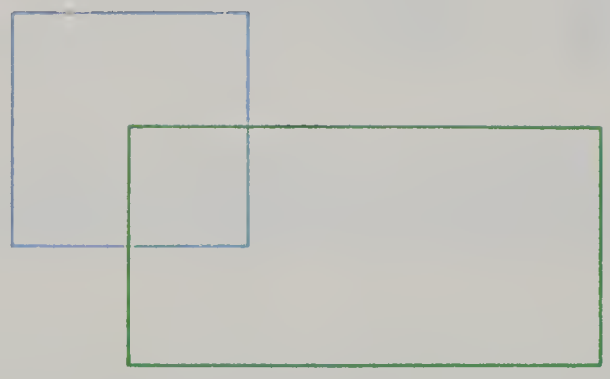
in A but not in B



On this page, the unit for measuring area is "1 square inch." The idea is suggested by the blue-tinted space in the diagram.

On this page, the unit for measuring perimeter is "1 inch." This idea is also suggested in the sketch at the left.

A ruler marked off in inches will help in this study.



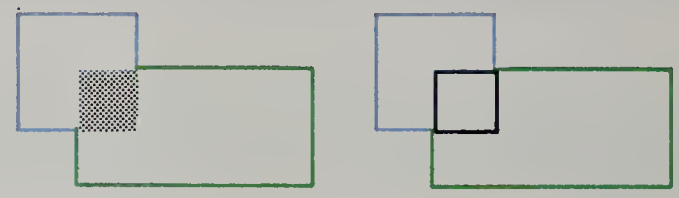
In this sketch, find the area and perimeter of each space indicated.

(spaces indicated)	Number of sq. in. in area	Number of inches in perimeter
in blue		
in green		
in blue <u>and</u> green		
in blue <u>or</u> green		
in green but not in blue		
in blue but not in green		
in blue <u>or</u> green, but not in blue <u>and</u> green		

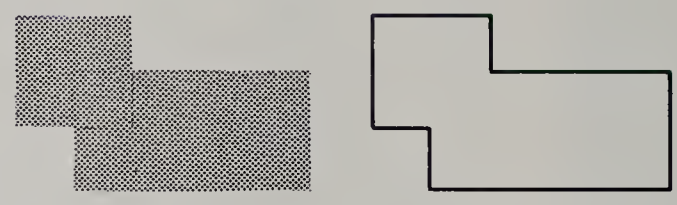
Definitions and hints:

Area of space indicated ↓

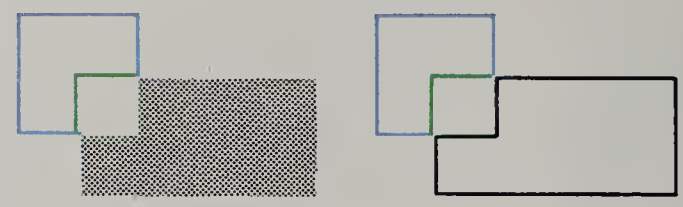
Perimeter of space indicated ↓



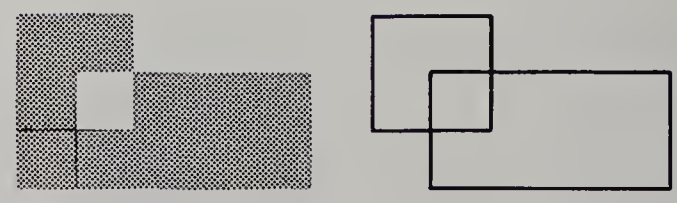
in blue <u>and</u> green



in blue <u>or</u> green

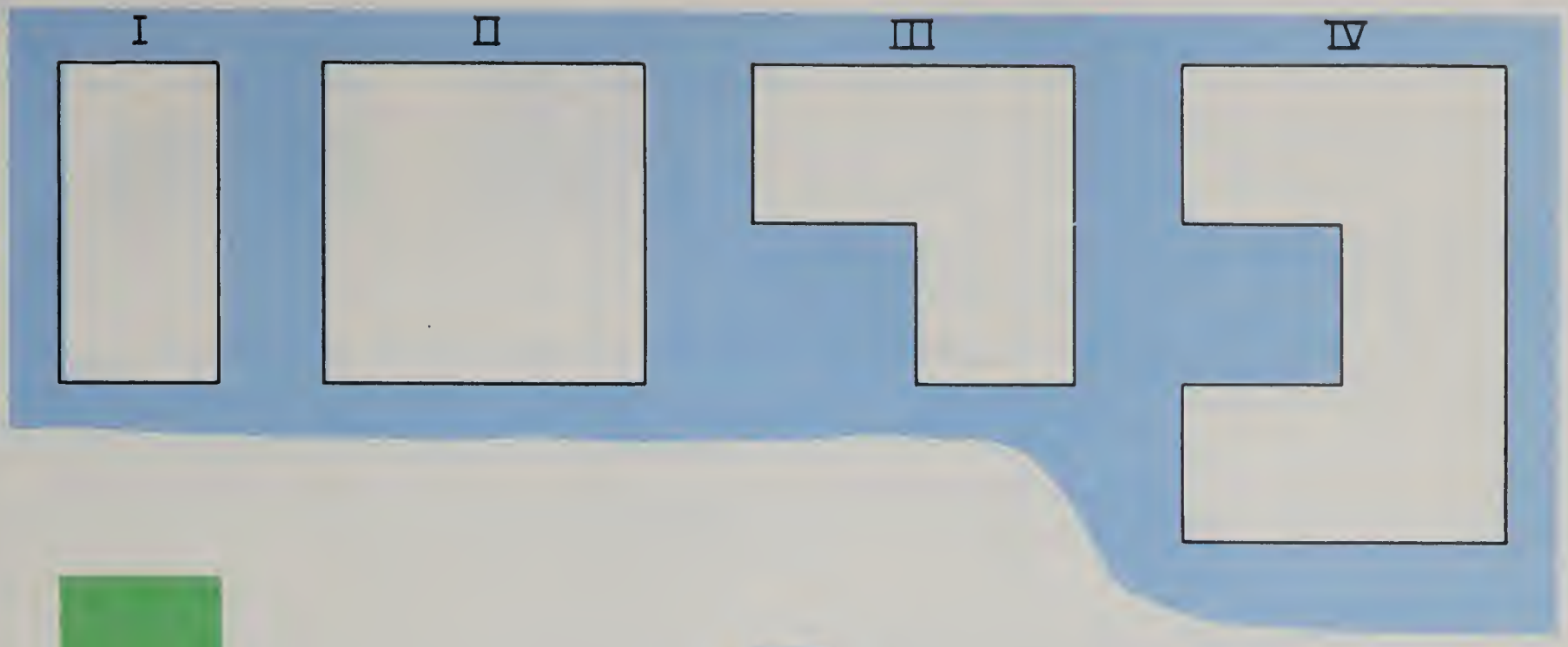


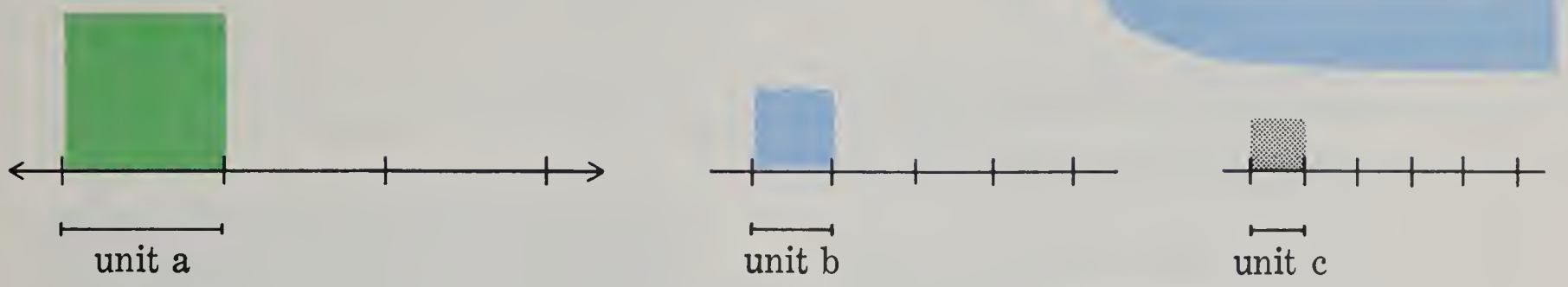
in green but not in blue



in blue <u>or</u> green, but not in blue <u>and</u> green

The area of this box is_____ square inches; its perimeter is_____ inches.

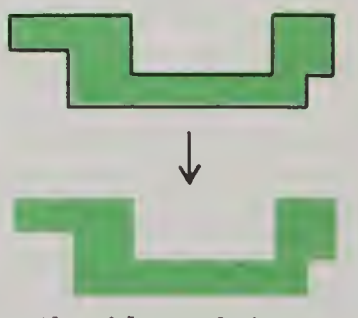




This chart suggests the idea of Area.	I	II	III	IV
How many green tiles are needed to cover each space above?				
How many blue tiles are needed to cover each space above?				
How many gray tiles are needed to cover each space above?				

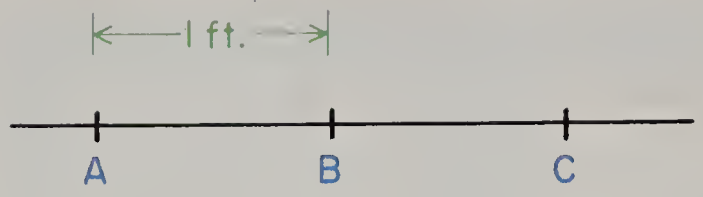
This chart suggests the idea of Perimeter.	I	II	III	IV
How many unit a's would it take to reach around the outside of each of the spaces above?				
How many unit b's would it take to reach around the outside of each of the spaces above?				
How many unit c's would it take to reach around the outside of each of the spaces above?				

In the sketch, ... the blue tint . . . . the black line
suggests the idea of Area
suggests the idea of Perimeter



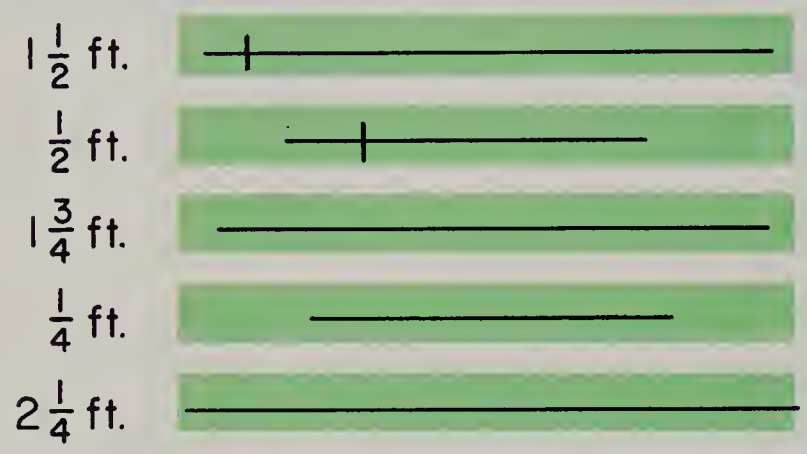
the idea of Area

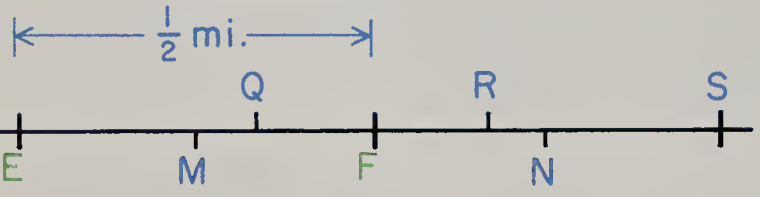
the idea of Perimeter



Let us agree that the lengths marked off above suggest 1 ft. (foot) each. Then the length from point A to point C suggests a length of 2 ft. (feet).

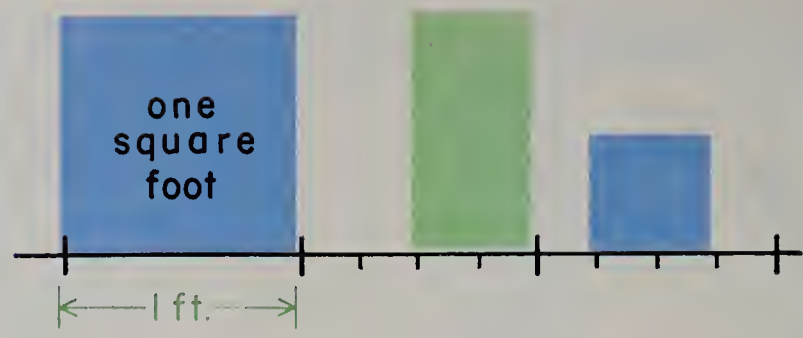
Mark off lengths on the lines below that would suggest the indicated length:





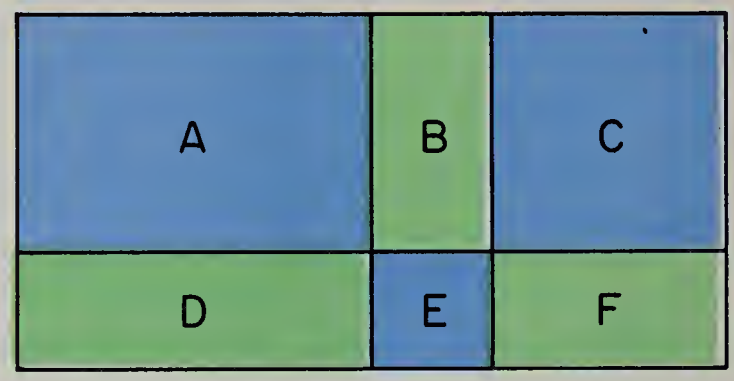
If the length of the line between points E and F suggests a half-mile, then what distances do the lengths indicated below suggest?

Length	Distance suggested
between E and S	1 mile
between E and M	
between R and S	
between M and S	
between E and R	
between Q and R	
between Q and F	
between M and Q	
between M and R	



If the length marked is to suggest 1 foot, then the large blue-tinted area suggests 1 square foot; the green area suggests $\frac{1}{2}$ of a square foot; and the small blue area suggests $\frac{1}{4}$ of a square foot.

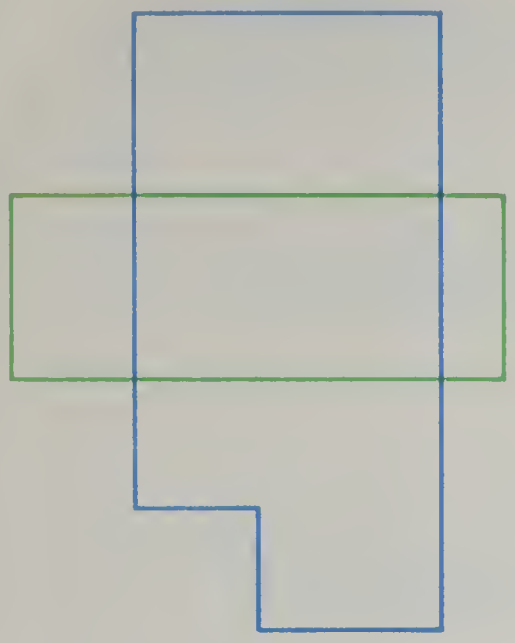
In the sketch below, indicate the area suggested by each section that is labeled.



Section	Area is
A	
B	
C	
D	
E	
F	

Find the perimeter of the sections.

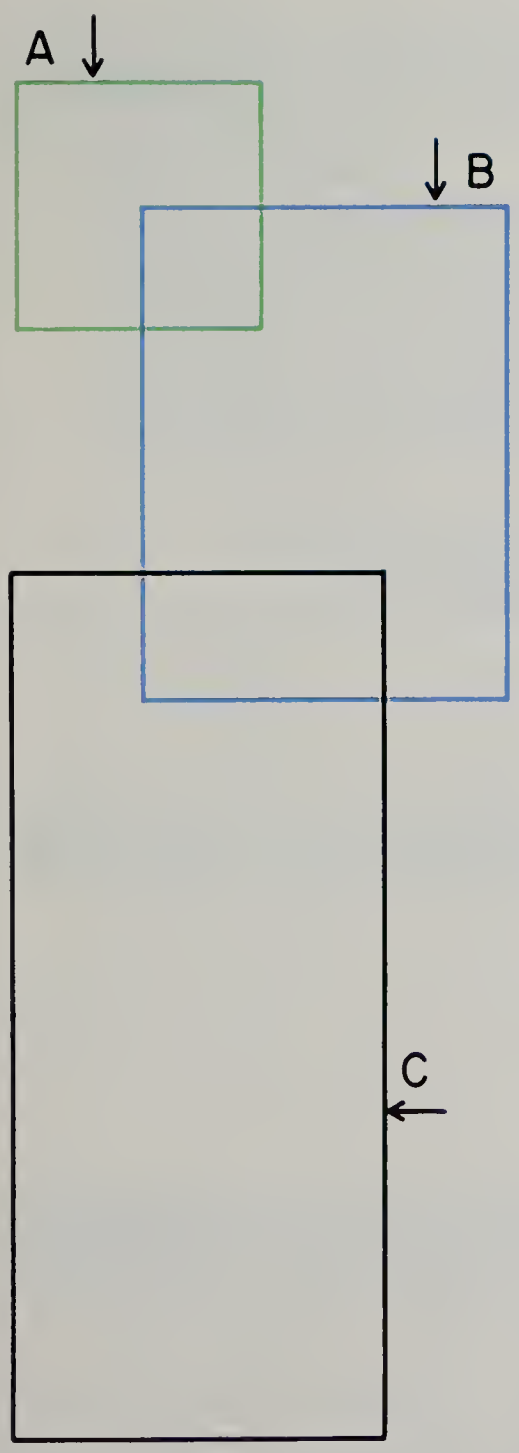
Section	Perimeter
A	
B	
C	
D	
E	
F	



The scale for this sketch is:

One-half inch represents one foot.

Area indications	Area	Perimeter
a. In G (green)	6 sq. ft.	11 ft.
b. In B (blue)	sq. ft.	ft.
c. In B and G		ft.
d. In B or G		
e. In B but not in G		
f. In G but not in B		



The scale for this sketch is:

One inch represents ten miles.

Area indications	Area	Perimeter
g. In A (green)	100 sq. mi.	40 mi.
h. In B (blue)		
i. In C (black)		
j. In A and B		
k. In B and C		
l. In A or B		
m. In B or C		
n. In A or C		
o. Not in B		
p. Not in C		
q. Not in A		
r. Not in B or C		
s. Not in A or B		
t. Not in A or C		

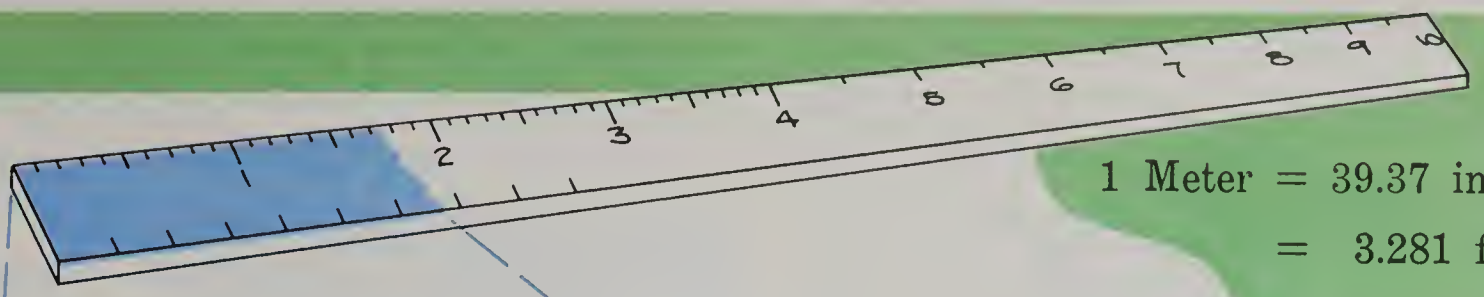
Complete each column based on the given information.

sum	5	16			13			7	10	17	16	
addends	2	9	10	7		4	5					factors
	3	7	50	12	5							factors
	6	63	500			44	35	12	21	30	39	product

sum + 10 (s +10)	20		22						20		21	
sum (s)	10			11		12	14	20		17		
addends	4	9	6		8		7					factors
	6	3	6	4								factors
	24				48	35		100	25			product (p)
	31	34		35						79	37	product + 7 (p + 7)

$x+y+9$	15	20										
$2\cdot(x+y)$	12								10		26	
$x+y$	6							$2\frac{1}{2}$				
x	2	5	4	1	$\frac{1}{2}$	$\frac{3}{4}$	.5			7		x
y	4	6	4	$\frac{1}{2}$	$\frac{1}{2}$	3	.5	2				y
	8									$10\frac{1}{2}$		$x\cdot y$ or xy
	23	45									57	$xy+15$
	16								12			$2\cdot xy$ or $2xy$

$m+n$			1.5	5.8		.6	18	1	1.5	9.9		5
m	.2	.4	.5	5	1.2						1.2	
n	.3	.6			9							
mn				4		.09	81	.25	.56	8.1	1.44	6.25



1 Meter = 39.37 inches
= 3.281 feet
= 1.094 yards

Scientists everywhere in the world and everyone in all large countries except the United States and the British Commonwealth, measure lengths or distances in meters . . . and kilometers (1000 meters), decimeters (one-tenth of a meter), millimeters (one-tenth of a decimeter, or one-hundredth of a meter).

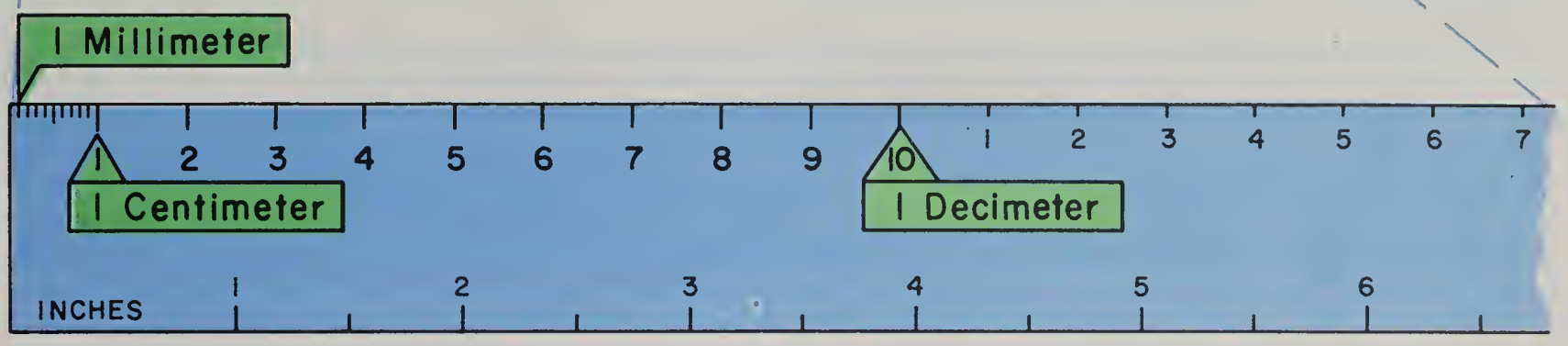
Since 1 meter is 39.37 inches long, we can show only a section of a meter stick in full size here.

U. S. Currency

1 dollar = 10 dimes
1 dime = 10 cents
1 cent = 10 mills

Metric System (distance)

1 meter = 10 decimeters
1 decimeter = 10 centimeters
1 centimeter = 10 millimeters



About 1790, the Academy of Sciences of Paris decided that there ought to be a standard (to measure distance) that the world could agree upon. After much consultation, they decided it ought to be

> . . . one ten-millionth of the distance from the North Pole to the Equator along the meridian that runs through Dunkirk, France, and Barcelona, Spain.

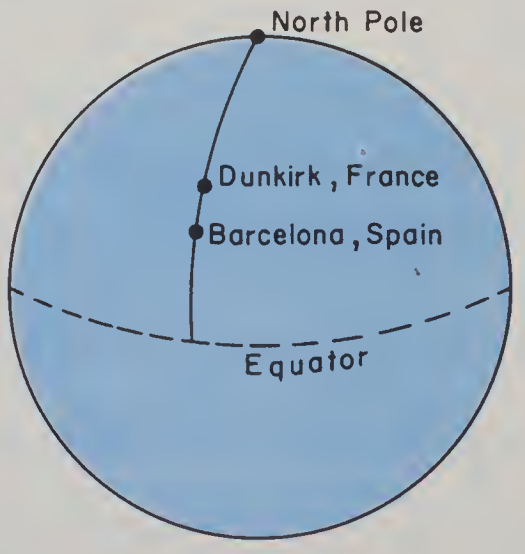
They computed that the distance between these cities was a little more than one-tenth of the total and, once it was measured, they could determine the distance from the North Pole to the Equator.

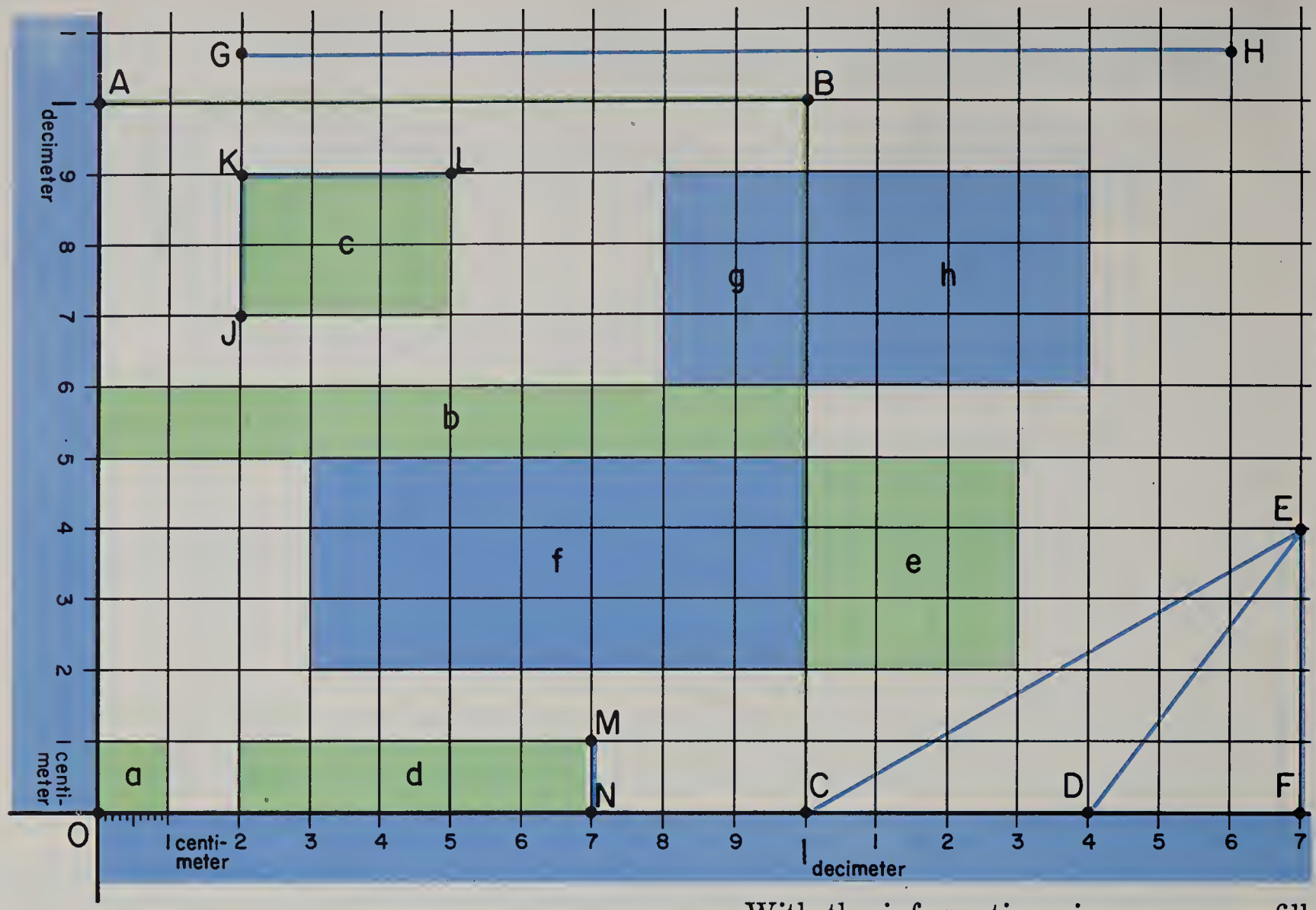
After many years work on the part of many mathematicians, engineers and surveyors, they thought they had determined the distance at sea-level to the best of their ability with the instruments then available.

On June 22, 1799, a platinum bar was adopted as the "true meter." It was placed in the French Archives of State.

In 1889, 31 copies of this bar were made as accurately as possible. The copy marked "No. 27" was sent to the United States.

In 1893, by Act of Congress, the "meter" was officially made the standard of measure for the United States. Unofficially, however, it was never adopted, and the people of the U.S. still use the yard as its standard.





Length

This diagram is full size. The line segment between points A and B is one decimeter long (one-tenth of a meter).

The line between points M and N is one centimeter long. The line between A and B is 10 centimeters long.

The distance between each small mark below the green square marked "a" is one millimeter — one-tenth of a centimeter.

Area

The small square marked "a" is one centimeter by one centimeter, and is called "one square centimeter."

The square whose corners are marked "A, B, C and O" is one decimeter by one decimeter, and the area inside the square is called "one square decimeter."

So, the area marked "b" is ten square centimeters, or one-tenth of a square decimeter.

With the information given, you can fill in the chart below, expressing the length of each line segment in meters.

Express the lengths of the line segments indicated in numbers of —

	meters	decimeters	centimeters	millimeters
M N	.01	.1	1	10
D F	.03		3	
A B				
G H				
D E				
C E				
C F				
B C				
J K				
N D				
N F				

Small green block "a" is one square centimeter.

The square with corners marked "A, B, C and O" is one square decimeter.

How many square centimeters are there in a square decimeter? _____

How many square decimeters are there in a square meter? _____

How many square millimeters are there in a square centimeter? _____

To summarize:

1 sq. meter = 100 sq. decimeters

1 sq. decimeter = 100 sq. centimeters

1 sq. centimeter = 100 sq. millimeters

With this information, you can fill in the chart below.

Express the areas of the indicated parts of the diagram as numbers of —

	square meters	square decimeters	square centimeters	square millimeters
a	.0001	.01	1	100
c			6	
d				
e				
b				
f				
h (in square) A,B,C,O				
a + d				
g				
g + h				
f + e				
f + b (in triangle) E,D,F,E				
(in triangle) E,C,F,E				
(in triangle) E,C,D,E				

Suppose the same drawing was a scale drawing—in which 1 centimeter in the drawing represented 1 inch.

Now express the line segments as numbers of —

	yards	feet	inches	16th of an inch
M N	$\frac{1}{36}$	$\frac{1}{12}$	1	16
D F	$\frac{1}{12}$	$\frac{1}{4}$	3	
A B			10	
G H			14	
C E				
C F				
J K				

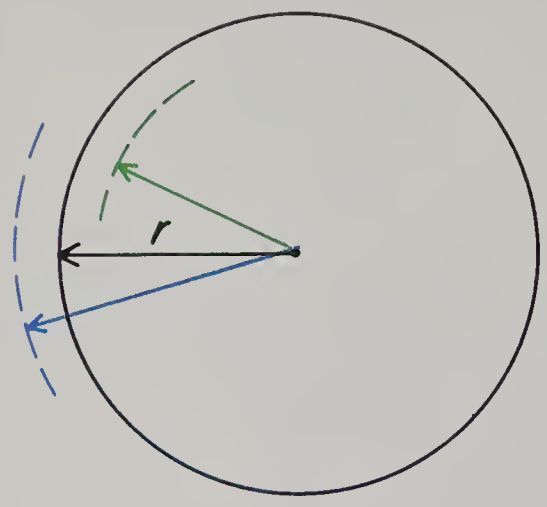
And, suppose that the same scale holds as we consider the area of sections of the diagram. The small green block marked "a" suggests 1 square inch.

Express the size of each area indicated as a number of—

	sq. yards	sq. feet	sq. inches	sq. 16th of an inch
a	$\frac{1}{1,296}$	$\frac{1}{144}$	1	256
b			10	
c			6	
e				
f				

Now you have an idea of why, in 1893, the U.S., by Act of Congress, adopted the meter as our official standard of measure.

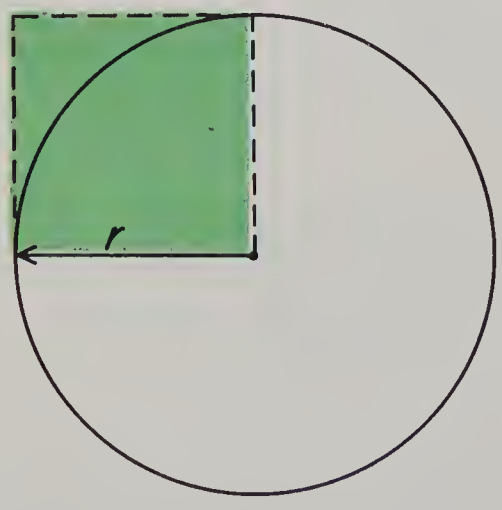
Why do you think we still use yards and feet?



A line drawn from the center to any point on a circle's perimeter is called a "radius;" a radius of the circle above is labeled "r."

If a circle had a shorter radius — such as the length of the green arrow — would its area be larger or smaller?

If its radius were longer than r suggested by the blue arrow — would the area of the circle be larger or smaller than the one in the sketch above with radius r?



How many tiles the size of r^2 ($r \cdot r$ or "r squared") would be needed to cover the circle and the area inside the circle?

Could you place two whole tiles with size r^2 inside the circle so they did not overlap and no point on the tile would be outside the circle? ____

How many such tiles would you need to be sure to cover all points on the circle and all inside the circle? _____

So, we see that the area of the circle is more than r^2 and less than $4r^2$.

That's not a very close approximation.

Suppose we use tiles that are $\frac{1}{2}r$ by $\frac{1}{2}r$ — or tiles with an area of $\frac{1}{4}r^2$?

Notice Diagram I on the next page. This is one way this might be done. The tiles inside the green line are clearly "inside" the circle. The tiles inside the blue line clearly cover all points on and inside the circle.

Put a dot in every tile that touches any point on the circle. Complete the blue line so it includes all tiles that touch the circle at any point.

How many tiles are there inside the blue line in Diagram I? _____

Now we know that the area is somewhere between 4 of these smaller tiles and 24 of them.

Too small: $4 \cdot \frac{1}{4}r^2$, or r^2

Too large: $24 \cdot \frac{1}{4}r^2$, or $6r^2$

This is a very poor approximation, but it is only the beginning.

Now, suppose we use still smaller tiles. Let's make each tile $\frac{1}{4}r$ by $\frac{1}{4}r$ — or $\frac{1}{16}r^2$.

Follow the same procedure as before in Diagram II.

How many tiles are inside the green line — tiles that do not touch any point on the circle? _____

How many tiles are used to cover all the points on and inside the circle? _____

Now we know that the area is somewhere between _____ of these still smaller tiles and _____ of them.

Too small: _____ $\cdot \frac{1}{16}r^2 =$ _____

Too large: _____ $\cdot \frac{1}{16}r^2 =$ _____

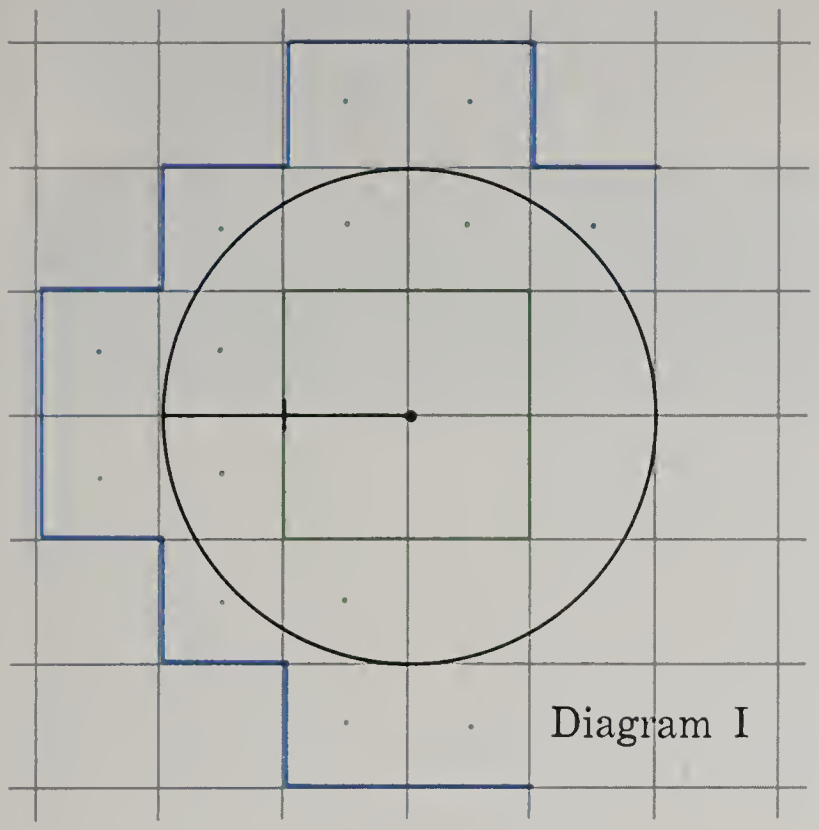


Diagram I

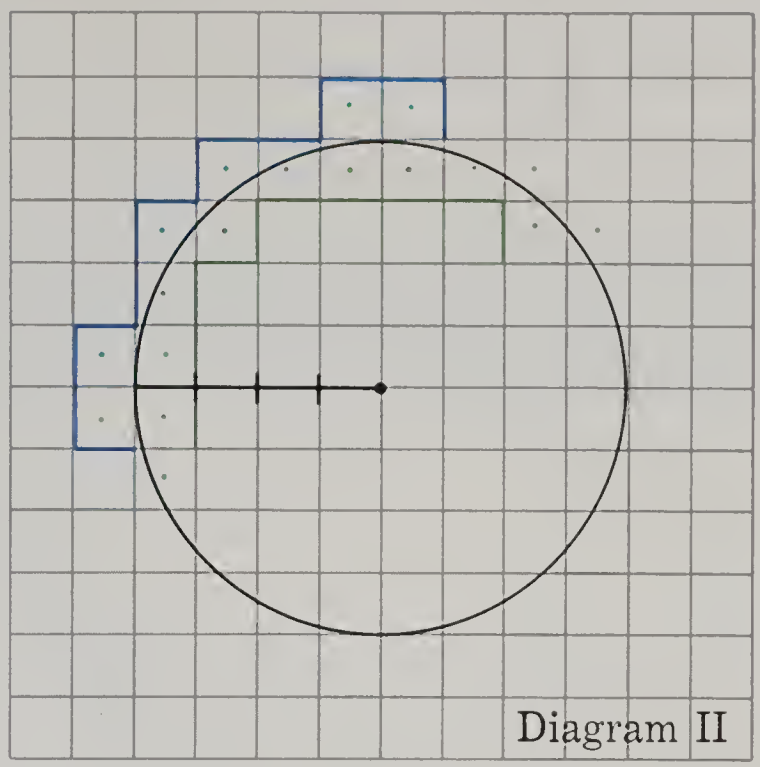


Diagram II

In Diagram III, we have tiles that are $\frac{1}{8}$ r on each side. So the tiles are

$\frac{1}{8}r \cdot \frac{1}{8}r =$ _____

How many tiles inside the green line? _____

Inside the blue line? _____

Too small: _____ $\cdot \frac{1}{64}r^2 =$ _____

Too large: _____ $\cdot \frac{1}{64}r^2 =$ _____

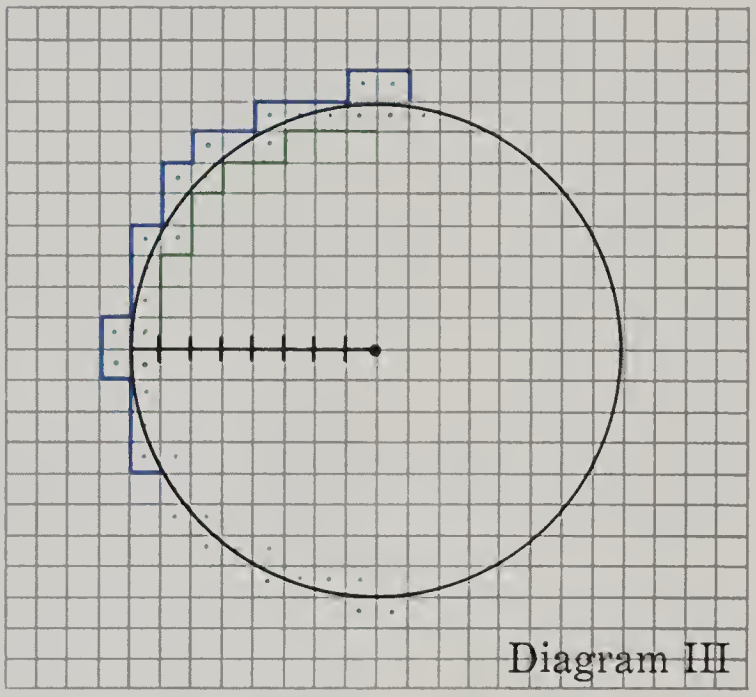


Diagram III

Now we need a magnifying glass. Diagram IV is seen as if it were twice size. (Only one-fourth of the diagram is shown. How can you know what the rest of the diagram would show?)

The radius is now divided into 16 parts, so each tile is $\frac{1}{16}r \cdot \frac{1}{16}r$ or _____

How many tiles inside the green line? _____

How many inside the blue line? _____

Too small: _____ $\cdot \frac{1}{256}r^2 =$ _____

Too large: _____ $\cdot \frac{1}{256}r^2 =$ _____

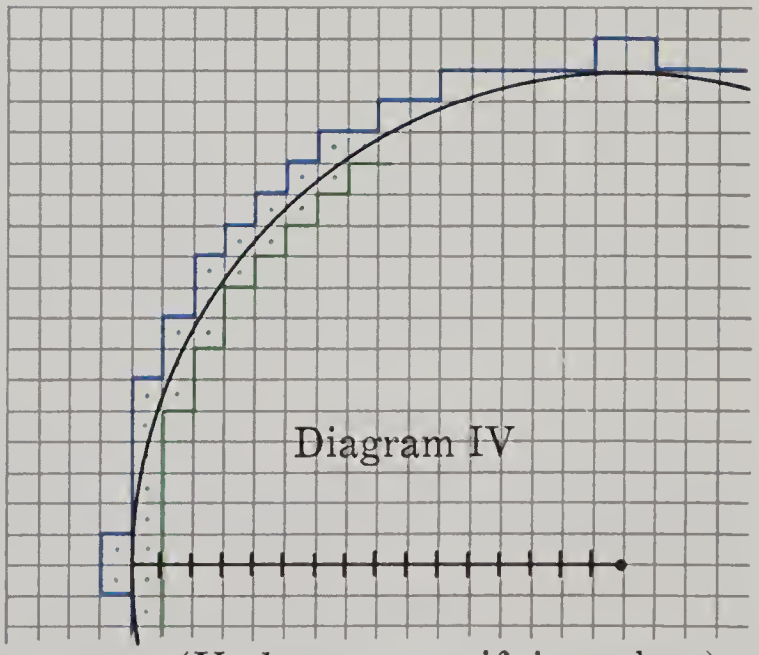


Diagram IV

(Under a magnifying glass)

Diagram	I	II	III	IV	V
Too small					
Too large					

(See the next page for Diagram V.)

Next we use a magnifying glass that makes things look five times as large as they are. Our radius now seems to be five times as long as it was in the first three diagrams.

We have made the tiles only $\frac{1}{40}$ of the length of the radius on each side, so the area of each tile is

$\frac{1}{40}r \cdot \frac{1}{40}r =$ _____

How many tiles are there inside the green lines? (You can find the number in the large blocks without drawing them.) There are _____ in this diagram of $\frac{1}{4}$ of a circle. In a diagram of the whole circle, there would be _____.

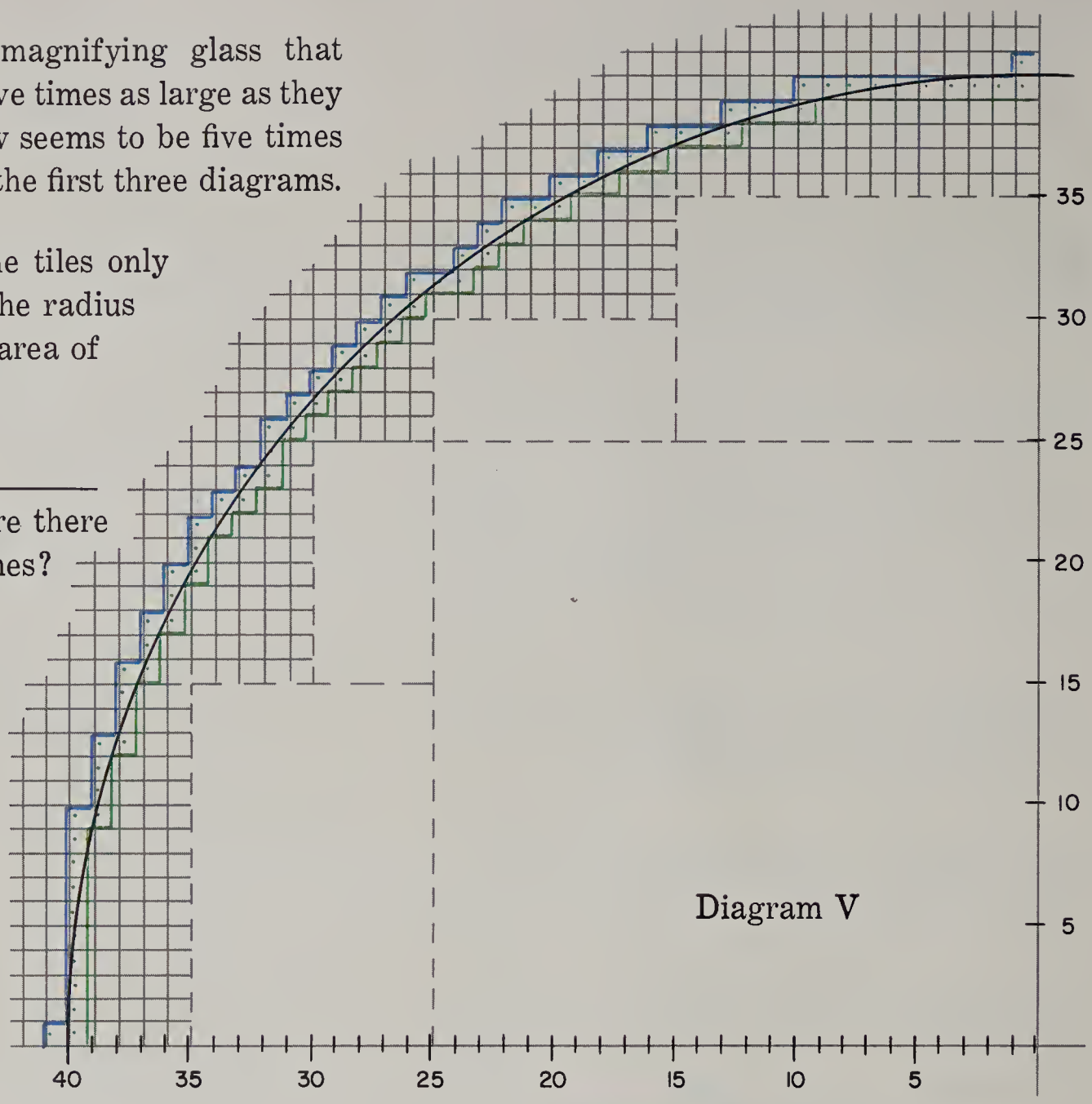


Diagram V

How many more than that are there inside the blue line in this fourth of the total diagram? _____ more. In a similar diagram showing the whole circle, there would be four times as many, or a total of _____ more blue than green. So, the total number inside the blue line would be _____.

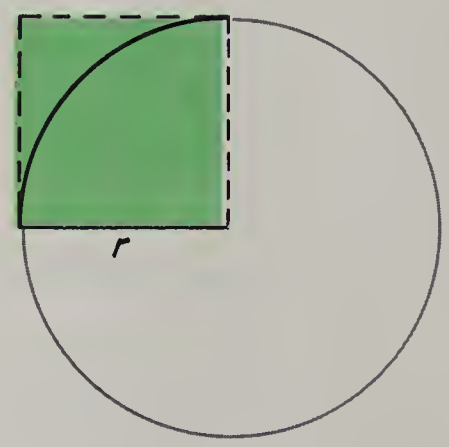
How many of these small tiles would be required to cover a square using the radius in Diagram V as the length of each side?

_____

Too small: _____ $\cdot \frac{1}{1600}r^2 =$ _____ r^2

Too large: _____ $\cdot \frac{1}{1600}r^2 =$ _____ r^2

(Record these results in the chart on the previous page.)



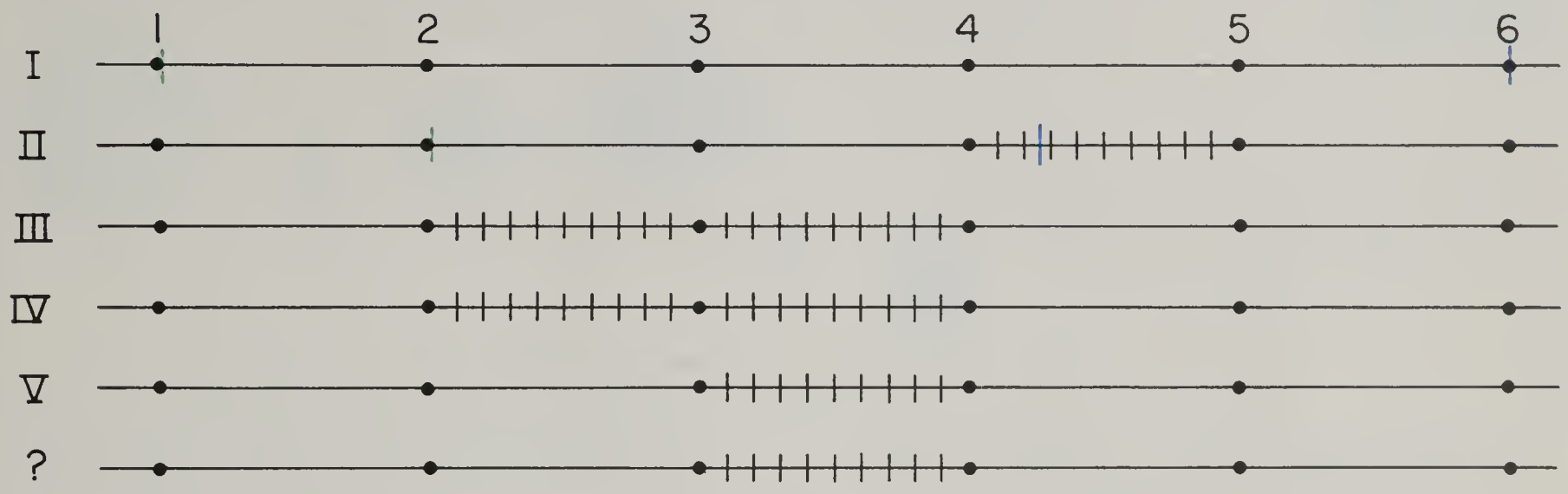
The green block above suggests our idea of r^2. If we cut four of these blocks into 1,600 tiny squares each, we know we could put at least _____ of the tiny squares completely inside our circle; and, we would not need more than _____ to cover every point on and inside the circle.

So, the area of our circle is somewhere in between.

Express the results of the five studies in decimals and record below . . . (to the nearest hundredth)

	I	II	III	IV	V	?
Too small	$1.0\,r^2$					
Too large	$6.0\,r^2$					

What do you notice? Now, indicate these results on a series of number lines.



We could keep magnifying our diagrams and divide the radius into the smallest units we could handle.

In each experiment, the difference between the area that is clearly "too small" and the area that is "too large" would become less and less.

Could we design an experiment with such a large radius and such small divisions that we would reduce that difference to zero?

_____

As we kept recording our continuously magnified experiments on a number line, would they crowd in from both directions on a certain point? _____

We know that such a point must be located somewhere between the points labeled _____ and _____. What point is halfway between? _____

Our experiments have led us to believe that the point 3.14 is very close to the number we're looking for.

By many similar experiments and by use of higher mathematics, we know that the point we are crowding is hard to name. It cannot be written as a fraction, using any numbers we know as numerator and denominator. We can name a point that is as close as we want — but that is all. We can write a decimal fraction as long as we like, but it never comes to a neat conclusion. The decimal fraction starts out

3.1416 . . . . . .

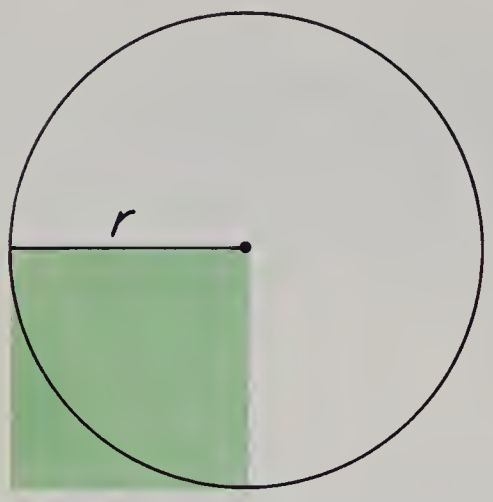
and those dots go on and on and on . . . .

We certainly came close.

For most calculations, that is close enough. Often we use $3\frac{1}{7}$ or $\frac{22}{7}$ as an approximation. What decimal fraction does that approximation give?

_____

3.14159265358979323846264338327950288419716939937510....= π



("r" is an abbreviation for radius. The green area suggests the idea of $r \cdot r$ or r^2 — read "r squared.")

How many tiles the size of a circle's radius squared would be needed to cover the inside of a circle? We found about 3.14 tiles would be needed.

About 300 B.C., some Egyptians found that 3.1416 was a more exact figure and in the year 1579 a Roman by the name of Vieta found that 3.1415926535 was even more exact.

With modern computors, even more exact estimates have been made, some to as many as 10,000 decimal places, the first 50 of which are given above.

This elusive number has been named by a letter of the Greek alphabet. The letter looks like this:

π

It is written "Pi" — and pronounced "pie."

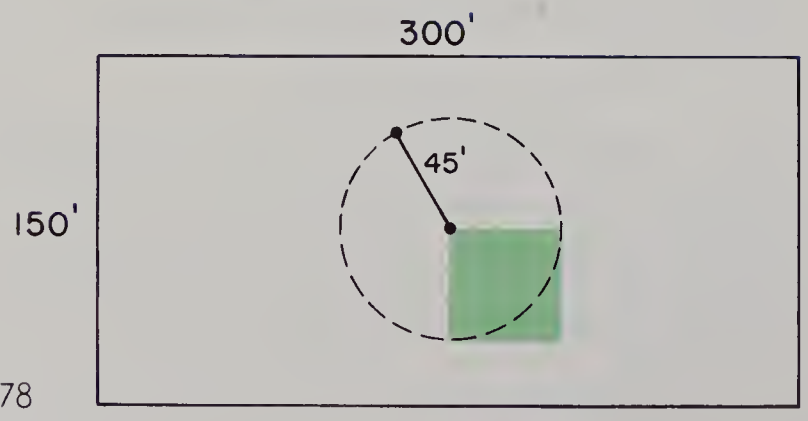
In making calculations, we use an approximation of π that suits our needs. Very often we use $3\frac{1}{7}$; when required, we use 3.1416. Astronomers and Engineers must use closer approximations.

You can get an idea of how close these approximations are by studying this problem:

Mr. Maxwell had a pasture 300 feet long by 150 feet wide. About in the center of this pasture was a post. Mr. Maxwell tied one end of a rope to this post, the other to a cow. Then he measured the rope and found that it was 45 feet long.

How many square feet of the pasture could the cow reach? . . . and how many square feet was it unable to reach?

A sketch may help:



We know that the area of the green square is 45' × 45' or _____sq. ft., and that the area the cow can reach is π times that amount. Now use the approximation we found (3.14) and those most often used ($3\frac{1}{7}$ and 3.1416) to find the area the cow could reach.

3.14 x _____ = _____

$3\frac{1}{7}$ x _____ = _____

3.1416 x _____ = _____

The greatest difference exists between the first and second approximation of the result — but they lead to conclusions that differ by only 6 square feet out of more than 6,000 square feet.

Try a still closer approximation — use π correct to 8 decimal places.

3.14159265 x 2025 = _____

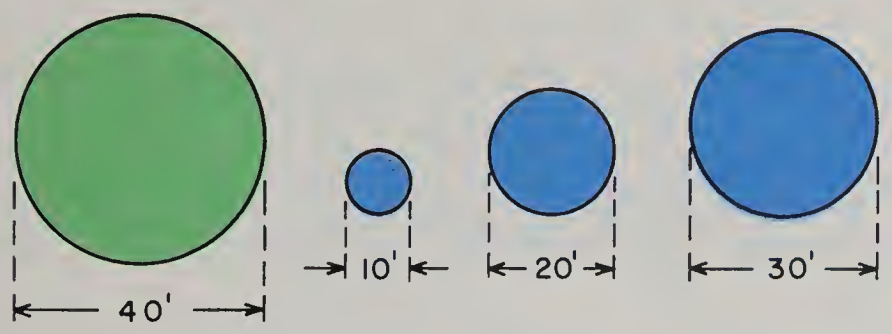
This is a difference of less than $\frac{1}{4}$ of a square foot out of more than 6,000 square feet.

We almost forgot — how many square feet of pasture was the cow unable to reach?

John Hayes, a gardener, said that he had in one day planted a round flower bed that was 40 feet across.

"That's nothing," Bill Jones said. "Today I planted three flower beds. One was 10 feet across; another was 20 feet across; and the third was 30 feet across."

John Hayes scratched his head. "Well, that's still not as much" . . . and an argument followed. Can you settle it?

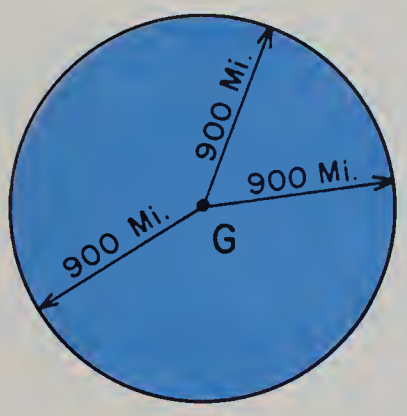


John Hayes planted _____ sq. ft.

Bill Jones planted _____ sq. ft.

(Use any of the approximations for π that we have discussed.)

When America's first Astronaut, Col. John Glenn, was in orbit in 1962, the newspapers reported that he and his capsule were often about 100 miles above the earth, and that he could see about 900 miles in all directions. Of course, mountains would sometimes cut off his vision and the curvature of the earth was involved; but we can make a rough estimate of the size of area he could see.



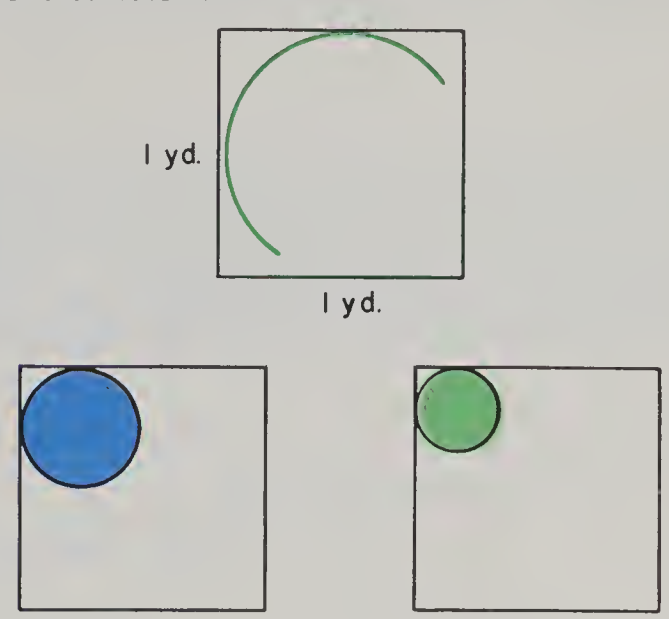
Col. Glenn could see about _____

_____

Mary said that if Pizza Pies were the same thickness, you would get the same amount in a pie that measured 14″ across as in four pies that measured 7″ across. Was she right? _____

Sarah had three pieces of cloth, each 1 yard square. From one, she wanted to cut the largest possible circle. From the second, she wanted to cut four circles the same size and as large as possible. From the third, she wanted to cut nine circles the same size and as large as possible.

Finish the sketches below that suggest the situation.



How much of Sarah's 3 square yards was used for the circles?

_____

In which case was there the most scrap cloth? _____

In the first case, when Sarah cut out only one large circle, how much of the 1 square yard piece was scrap?

_____

_____

In which case was there the least scrap?

_____

How much scrap was there altogether?

_____

If Sarah had another piece of cloth 1 yard square and wanted two circles the same size and as large as possible, what would the area of each circle be?

_____

How much would be scrap? _____

Explore the problem Sarah would face if she wanted to cut three circles as large as possible and all the same size from 1 square yard of cloth.

Use the given information to complete each column.

A

c	15	21		.5	5	$\frac{1}{4}$	.5		.1
$c + d$	75		29			1		105	
$c \cdot d$ or cd			100		.35			500	1
d	60	19		.7			$\frac{1}{2}$		

B

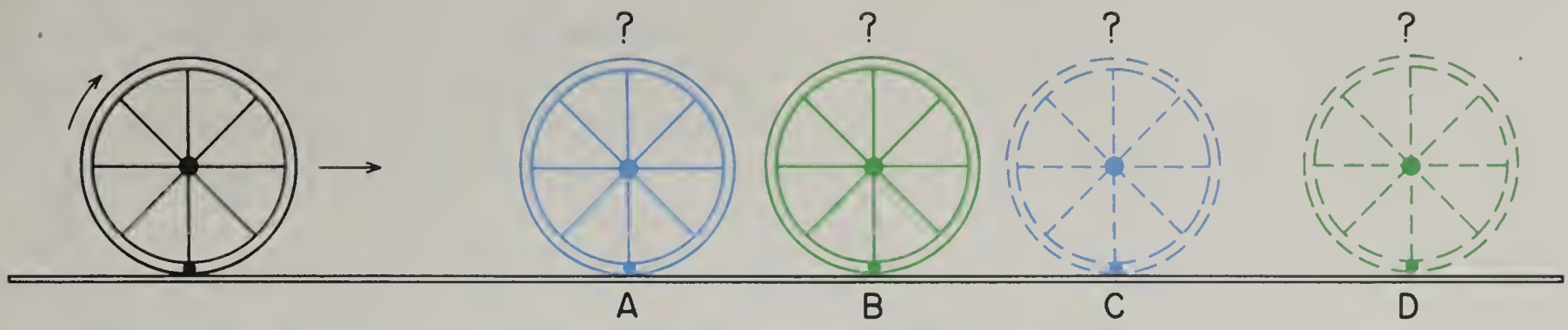
s	5		3	.7	$\frac{1}{2}$				
t	7			.9		$1\frac{1}{2}$	35	37	
$s + t$		24			$1\frac{1}{4}$				
st		144	105			$2\frac{1}{4}$			
$2 \cdot st$ or $2st$	70						2,450	3,182	
$2s + 2t$									256
$2s + t$	17					$4\frac{1}{2}$			253

C

x	25		5		$\frac{1}{4}$			.87	
xy		1.26			$\frac{1}{16}$		1.35		
y	5	.7		.3			.15	.1	
$2x + y$			11	.9		39			1,005
$2x - y$						5			995

D

$a \cdot b$ or ab	6.8	.49	1.44	2.56	13.5	2.10	1	10	.01
a			1.2	8	1.5	.6		.1	
b	2	.7					10		.1
$a + b$									

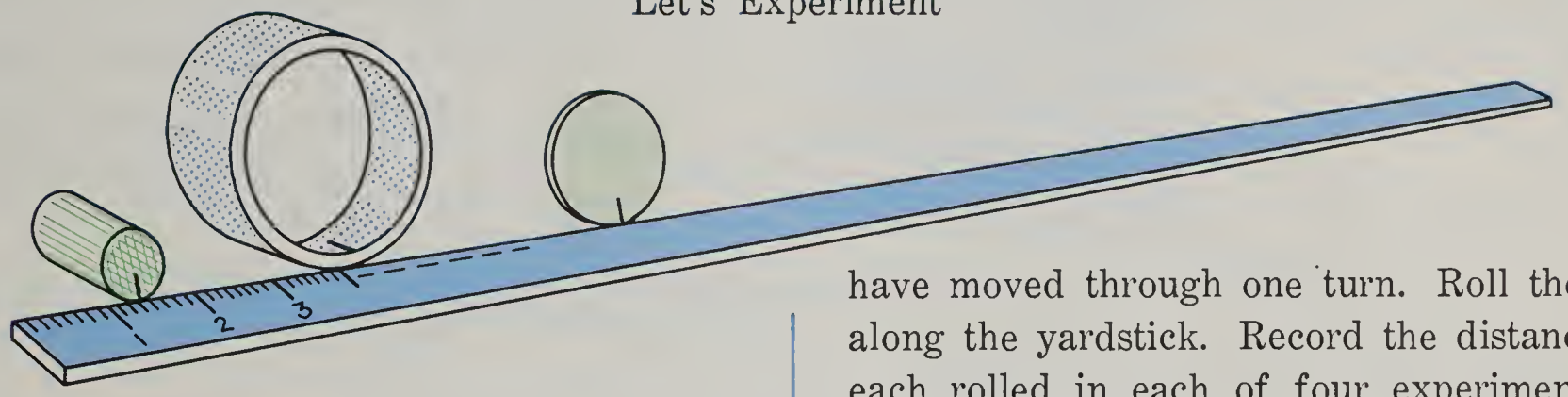


How far does a wheel roll in one turn?

Does a roller skate wheel go as far in one turn as a bicycle wheel? Which goes further in one turn—the large front wheel of a tricycle or its smaller back wheels?

In the diagram above, imagine that the black wheel rolls along to the right. It has a spot on it so we can tell when it has completed one turn. Which colored sketch indicates its position after one turn: A, B, C, or D?

Let's Experiment



You will need a yardstick and a pin.

Out of cardboard, cut wheels with diameters of 2″, 4″ and 8″. Near their outside edges, put a mark so you can tell when they have moved through one turn. Roll them along the yardstick. Record the distances each rolled in each of four experiments. Add the results and divide by four to find "average distance."

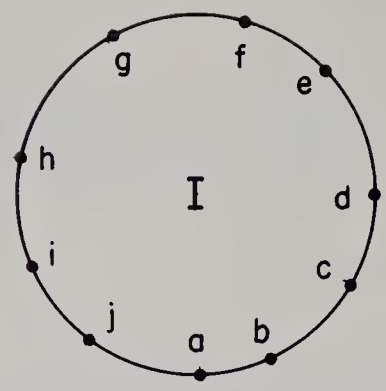
Continue experimenting with other objects such as tin cans, rolls of tape, wheels—anything that is round.

Name of object	Diameter of object	Distance traveled in 1 turn 1st exp.	2nd.	3rd.	4th.	Average distance traveled	Average distance = ____ X Diameter
Cardboard wheel	2"						____ = ____ X ____
Cardboard wheel	4"						____ = ____ X ____
Cardboard wheel	8"						____ = ____ X ____
							____ = ____ X ____
							____ = ____ X ____
							____ = ____ X ____
							____ = ____ X ____
							____ = ____ X ____

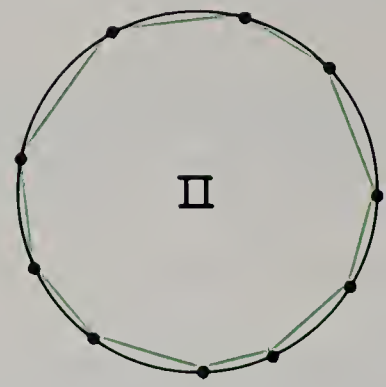
Add the 8 numbers written above the blue box and divide by 8 to find their average.

Another Experiment.

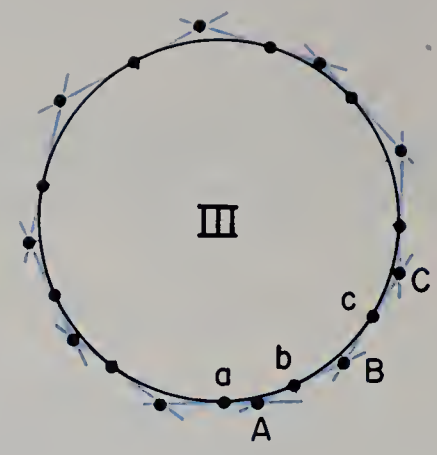
Draw a circle any size. Mark some points on the circle, any distance apart — they need not be evenly spaced (see I).



Connect these points with lines (see II).



Draw lines outside the circle that touch it only at the points you have already indicated. Where they cross each other, locate more points (see III.)

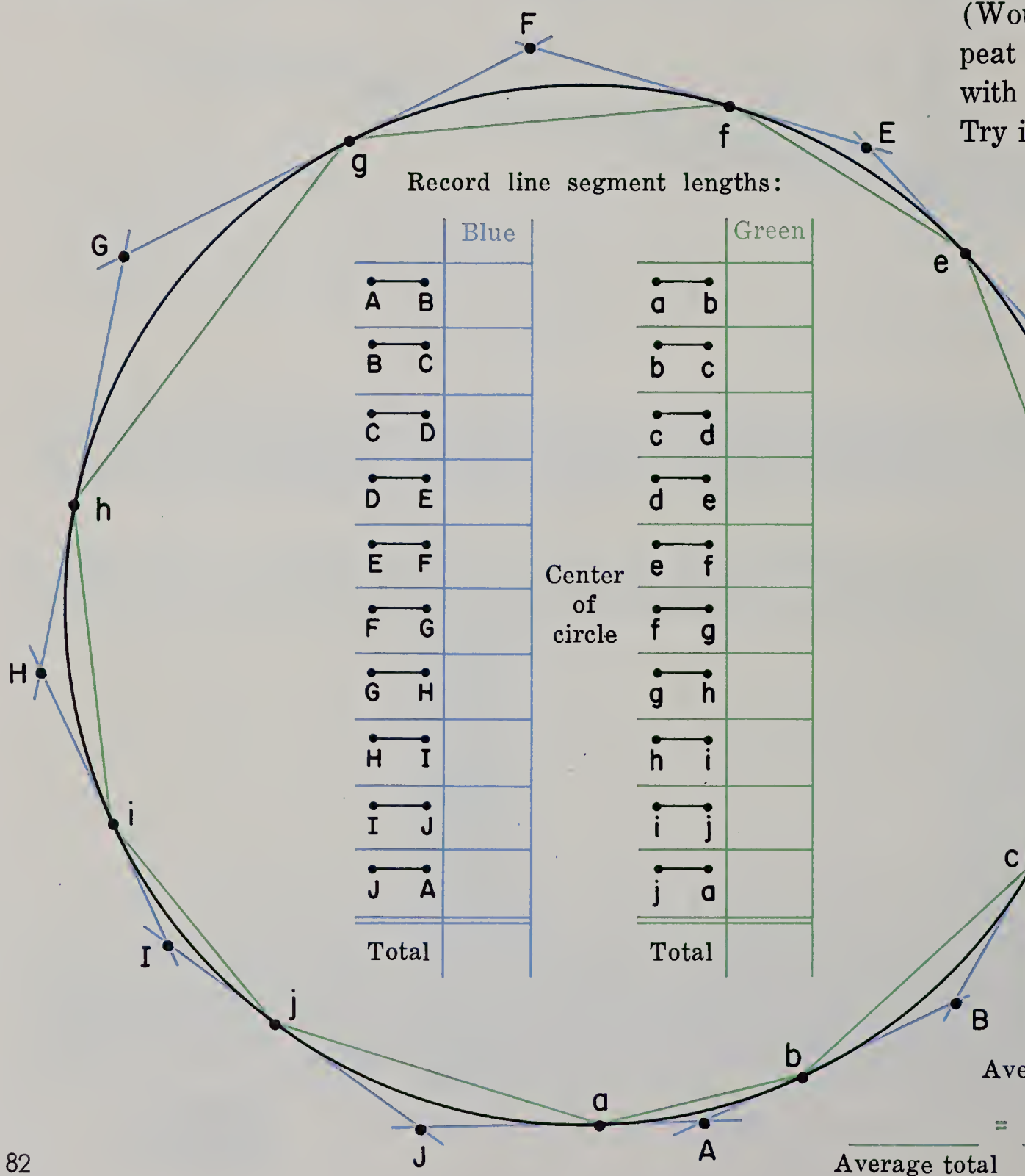


(Below is a single, enlarged drawing that we will use.)

Measure each green line segment and add the results together. Using the second group of points, measure each blue line segment and add the results.

Why must the distance around the circle — called its "circumference" — be somewhere between these two totals?

(Would you like to repeat this experiment with a larger circle? Try it.)



Record line segment lengths:

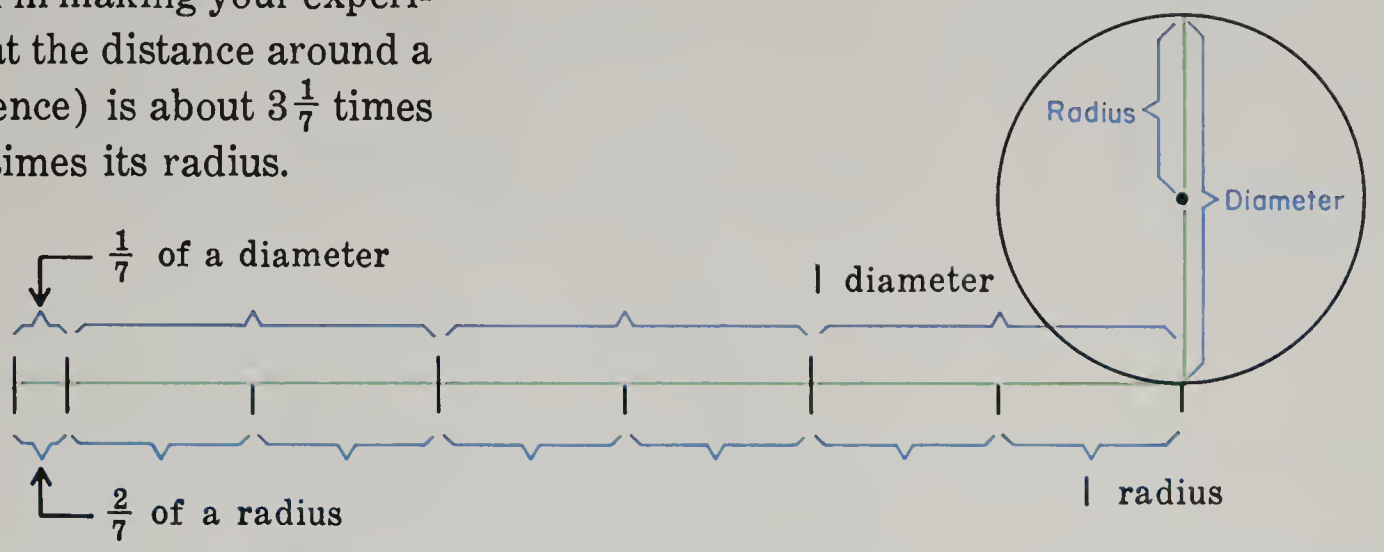
Blue		Green	
A B		a b	
B C		b c	
C D		c d	
D E		d e	
E F		e f	
F G		f g	
G H		g h	
H I		h i	
I J		i j	
J A		j a	
Total		Total	

Center of circle

Total of Blue	
Total of Green	
Average of totals	

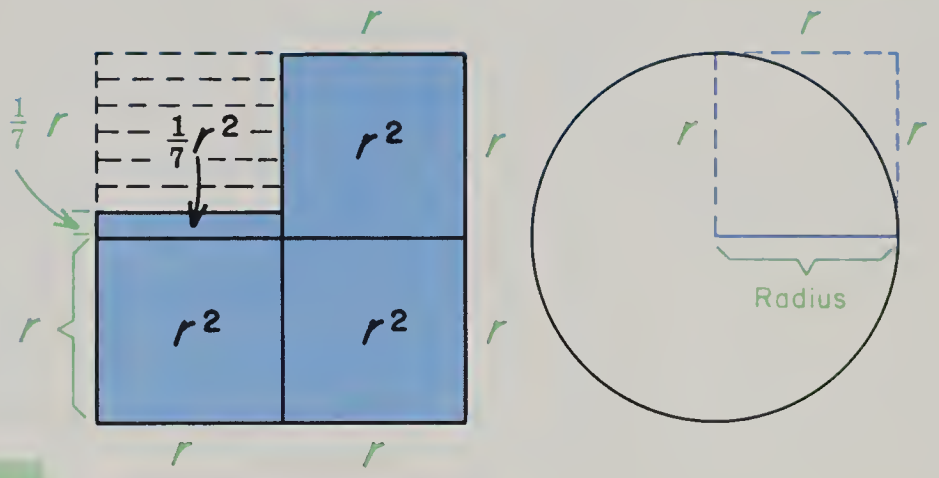
$$\frac{\text{Average total}}{\text{Diameter}} = \frac{\quad}{\quad} \times \frac{\quad}{\quad}$$

If you were careful in making your experiments, you found that the distance around a circle (its circumference) is about $3\frac{1}{7}$ times its diameter, or $6\frac{2}{7}$ times its radius.



That "about $3\frac{1}{7}$" sounds familiar. You found earlier that it takes about $3\frac{1}{7}$ squares, measuring the same on each side as the radius, to cover the area inside the circle.

You will remember that $3\frac{1}{7}$ is an approximation for a never-ending decimal:



3.1416 5926535..... = $\tilde{\pi}$ (written "Pi")

Where we are speaking about circles, there is some shorthand you will meet very often.

A is shorthand for "area"
d is shorthand for "diameter"
r is shorthand for "radius"
c is shorthand for "circumference"
$\tilde{\pi}$ is a name for the number that is approximately $3\frac{1}{7}$ or 3.1416 etc.

To find the area of a circle, we multiply $\tilde{\pi}$ times the radius times the radius. In shorthand:

$$A = \tilde{\pi} \cdot r \cdot r = \tilde{\pi} r^2$$

To find the circumference of a circle, we multiply the diameter by $\tilde{\pi}$. In shorthand, this is:

$$c = \tilde{\pi} d$$

Why can we find the circumference by multiplying the radius by $2 \cdot \tilde{\pi}$?

$$c = 2 \cdot \tilde{\pi} \cdot r = 2\tilde{\pi} r$$

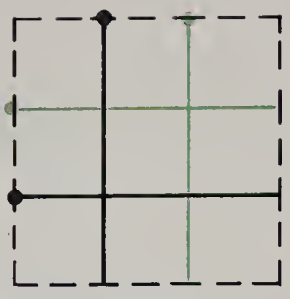
Complete the following chart of information about some circles. (It will be convenient to use $3\frac{1}{7}$ or $\frac{22}{7}$ as an approximation for $\tilde{\pi}$.)

Length of Radius	Length of Diameter	Area of Circle	Circumference of Circle
$\frac{1}{2}$"			
	2"		
2"			
	6"		
4"			
	10"		
7"			
	20"		

Is it true that you can find the area of the circles indicated above by finding $\frac{1}{2}$ of the radius times the circumference?

$$A \stackrel{?}{=} \frac{1}{2} rc$$

I a

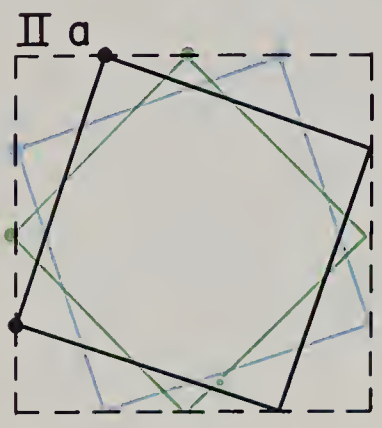


I b

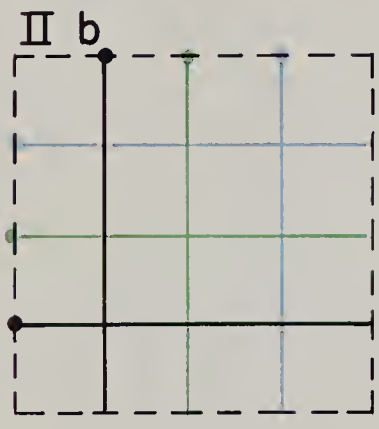
How Many Tiles?

On this page, we refer to tiles that are $\frac{3}{8}$" by $\frac{3}{8}$"

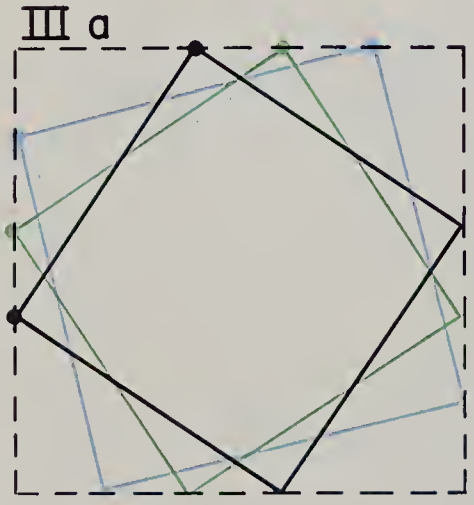
Complete the chart →



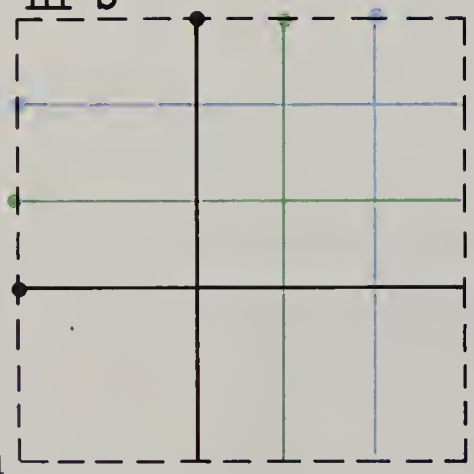
II a



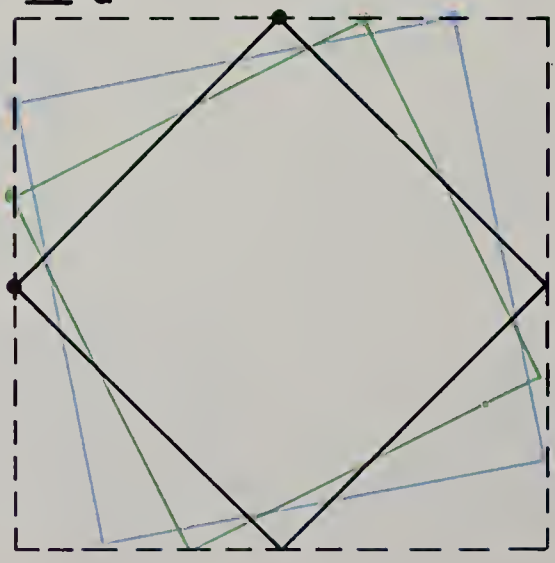
II b



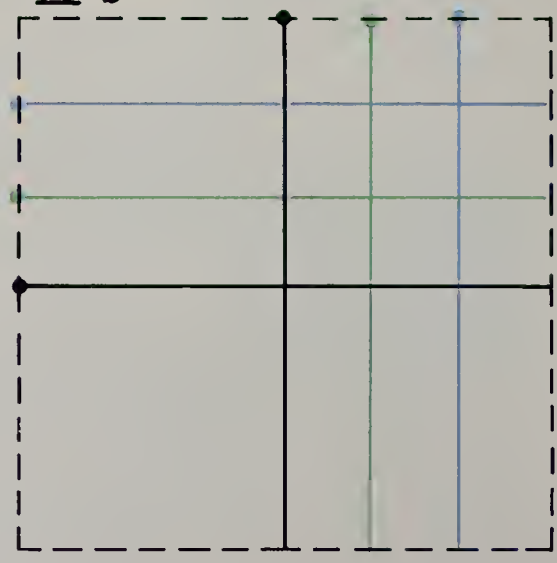
III a



III b



IV a



IV b

After this symbol, indicate the number of tiles that would fit inside the broken black line. (In examples marked "a.")

means how many tiles are required to cover the area outside the solid black or green or blue squares.

means how many tiles are required to cover area inside black, green or blue squares.

means how many tiles needed to cover area above and to the left of black, green and blue lines in examples marked "b."

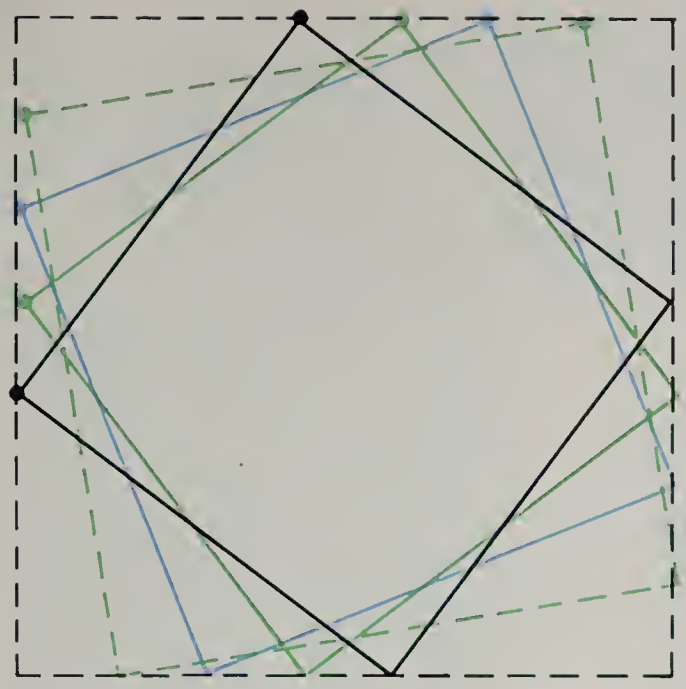
above and to the right.

below and to the right.

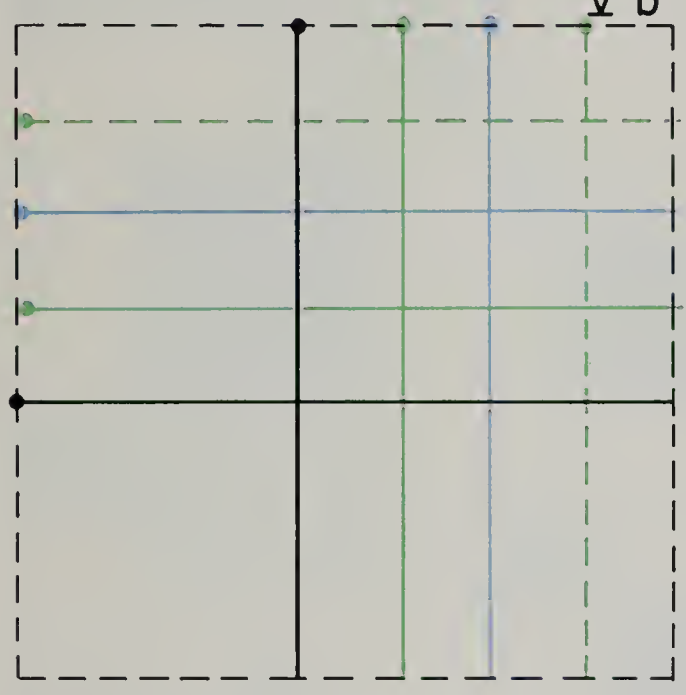
below and to the left.

(All numbers indicated mean "number of tiles.")

	I	II	III	IV
⬚	9	16		
◇•	4	6		
•+	2	3	6	
+•	2			
◇•	5			
+•	1		4	
+••	4			
◇•				
•+				
+•				
◇•				
+•				
•+				
◇•				
•+				
+•				
•+				

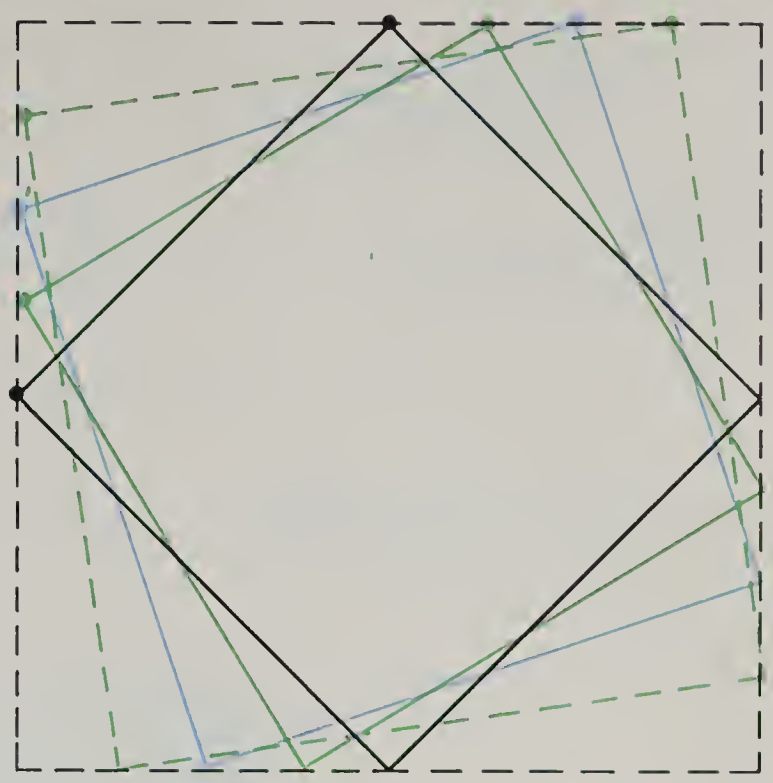


V a

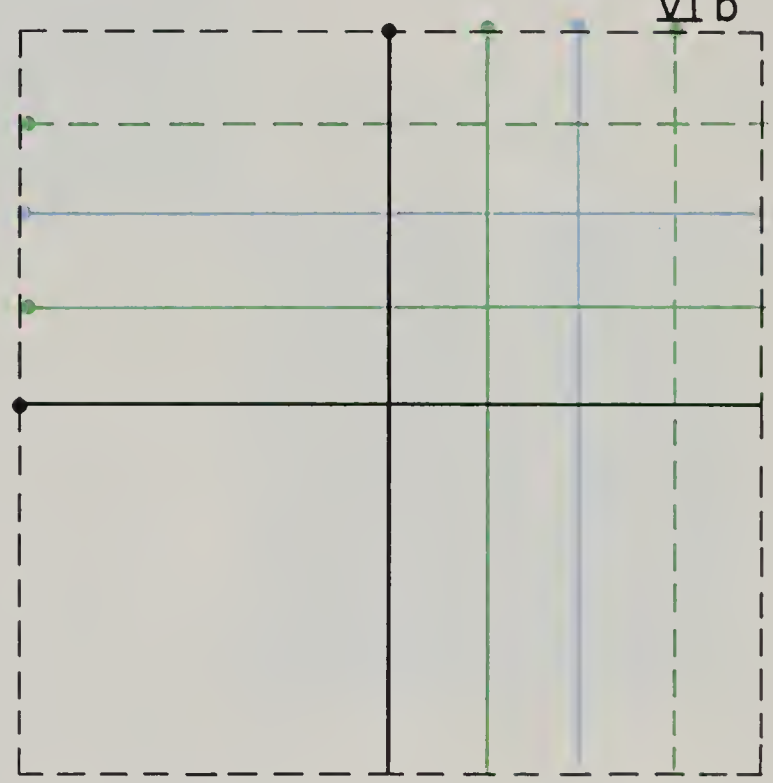


V b

How many tiles?



VI a



VI b

Keep your eyes open for patterns!

Complete the blue-line chart below, following agreements from previous page.

V	49												
VI													

In the record below, summarize indicated results from the previous page.

I	9	5											
II	16	10											
III	25	13											
IV	36	18											

In each group of three results, does the result written on the left have any relationship with the two written at its right?

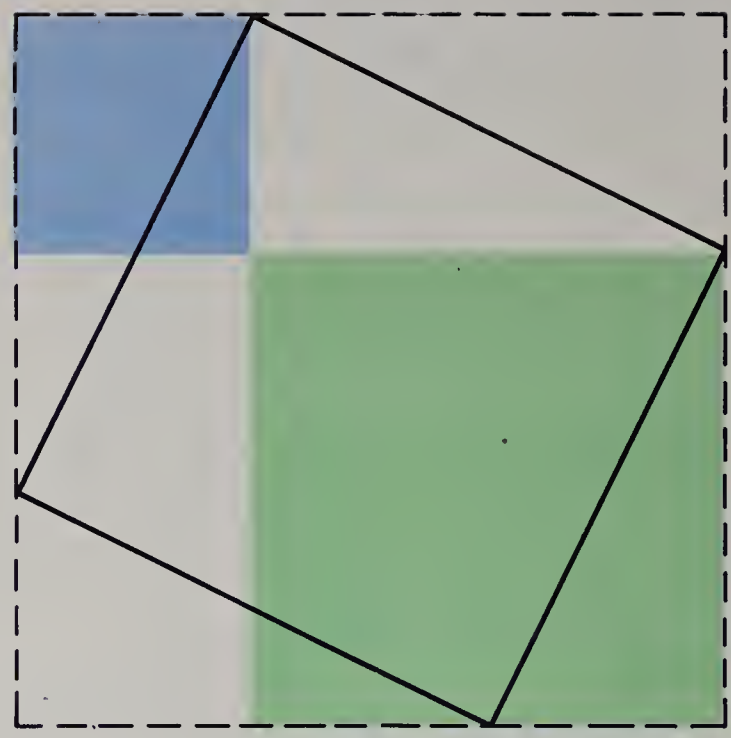
You will need a ruler! Find three results to answer these questions about each diagram:

a. How many square inches are there inside the heavy, solid black line?

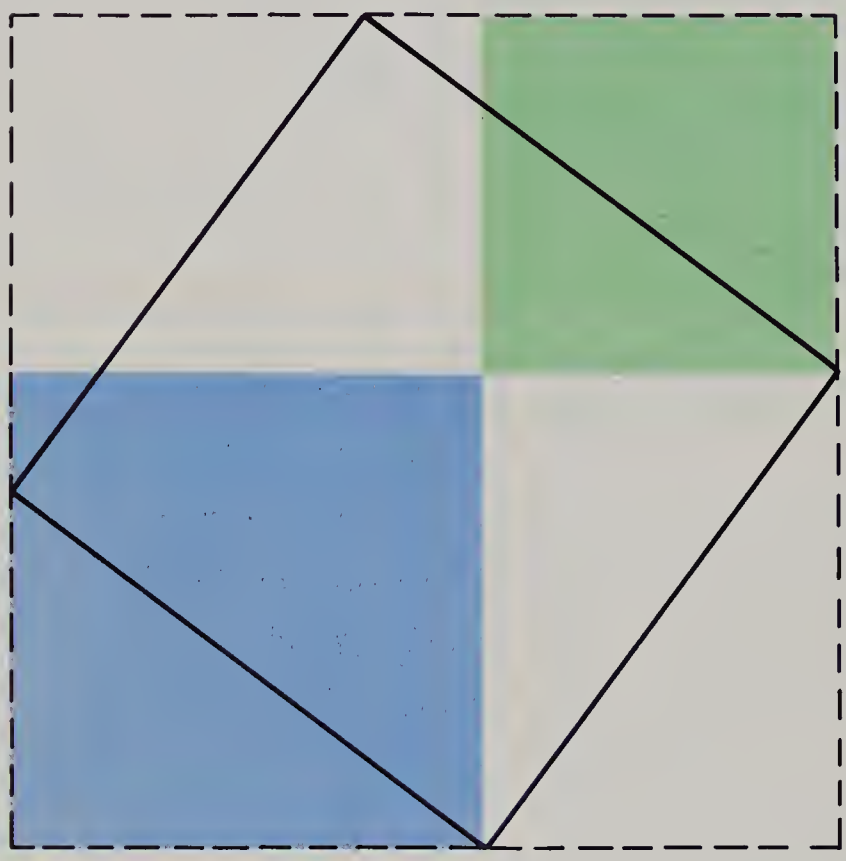
b. How many square inches in the part tinted blue?

c. How many square inches in the part tinted green?

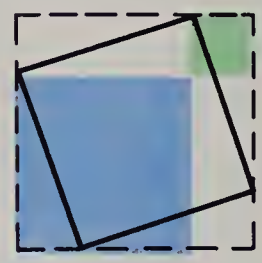
a



b

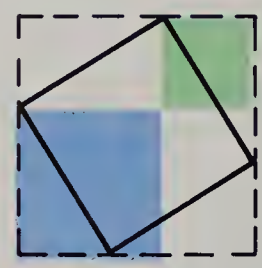


c

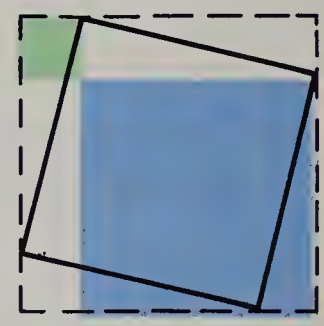


d

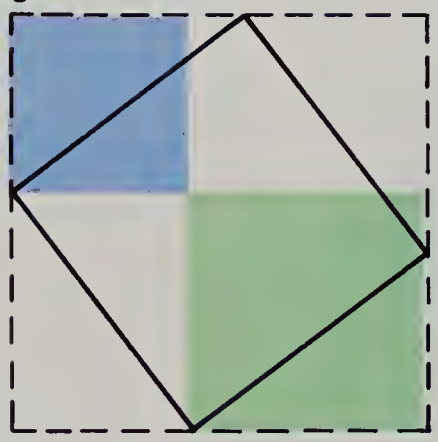
e



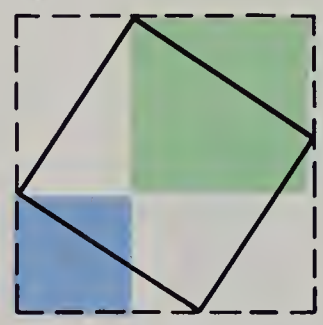
f



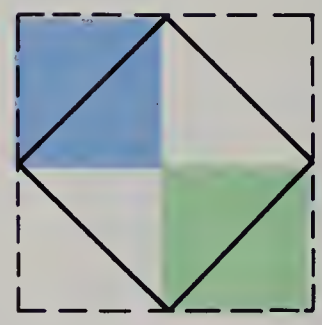
g



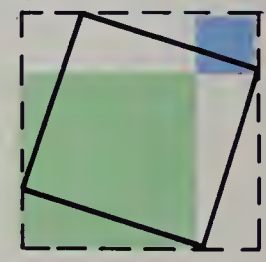
h



i

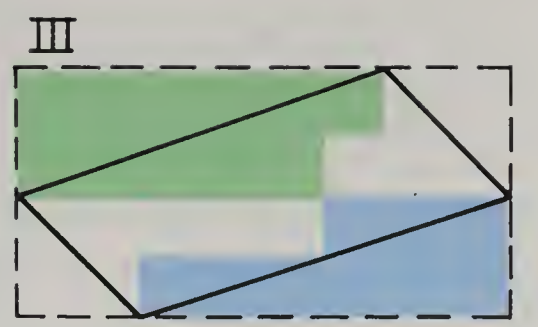


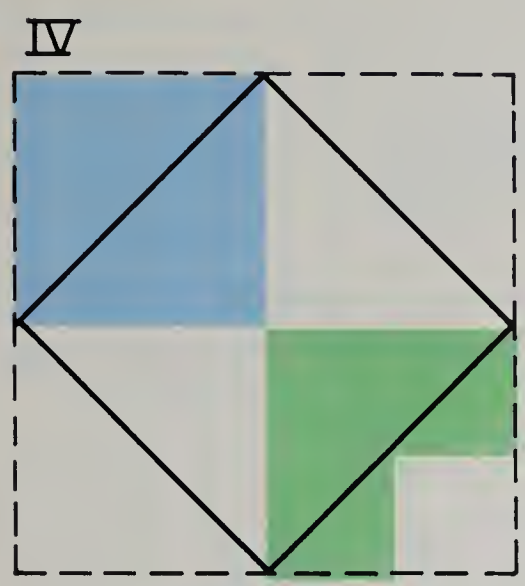
j

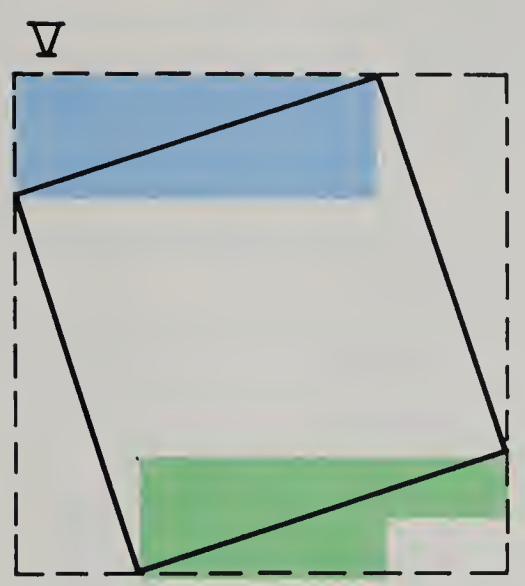


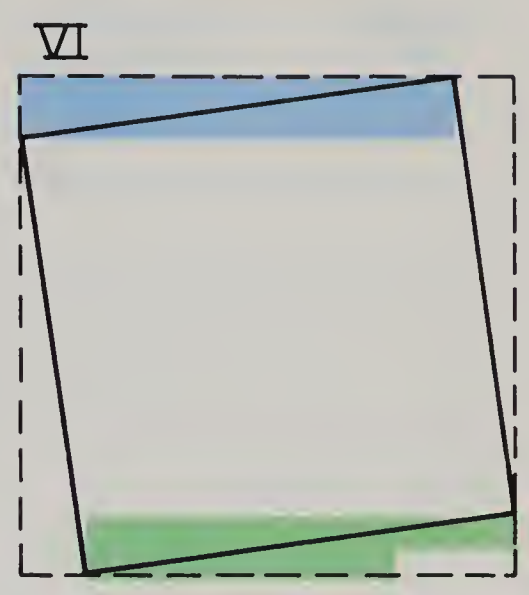
Make a record of the number of square inches in the indicated areas.

Indicated area	a	b	c	d	e	f	g	h	i	j
in green	4									
in blue	1									
inside black lines ◇		$6\frac{1}{4}$								









one square inch

On the left is a green block. The area is one square inch.

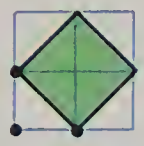
Using a ruler, measure the lengths of lines in the diagrams above that will answer the question in the chart below. (You may skip those parts in the chart that are tinted green if you want to.)

Record numbers of square inches in the areas described

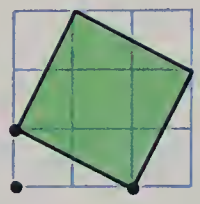
	I	II	III	IV	V	VI
a. inside the broken black line						
b. blue						
c. green						
d. blue outside solid black line						
e. green outside solid black line						
f. blue inside solid black line						
g. green inside solid black line						
h. inside solid line, but not green or blue						
i. blue or green outside solid line						
j. inside solid line						
k. inside broken line, but not inside solid line						

On this and the following page we are going to experiment with square arrays of tile that are $\frac{1}{4}''$ by $\frac{1}{4}''$. Results will be recorded in the chart on the next page.

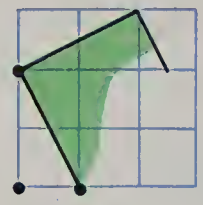
I



II



III



In a 2×2 array (I), a dot is marked in the lower left-hand corner. Mark two more dots — one, the width of a tile to the right and another, the width of a tile up. Connect these two dots.

Consider this line as the side of a square. How many tiles would be needed to cover the area inside? _____

Next, consider a 3×3 array (II). From a dot in the lower left-hand corner, locate two dots — two to the right and one up. Connect them, and consider the line as the side of a square. It will require _____ tiles to cover it.

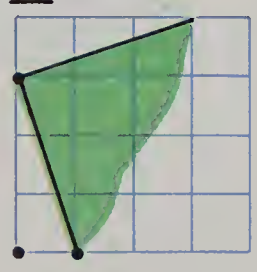
In diagram III, follow the same procedure, except that the two dots are to be located the width of one tile to the right and two tiles up.

Finish drawing the square. How many tiles are required to cover it? _____

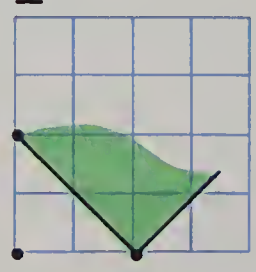
Did you expect the results in examples II and III to be the same?

Complete the following examples of 4×4 arrays:

IV



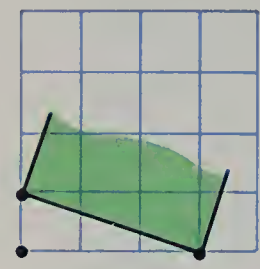
V



How many tiles are required to cover the squares drawn in the two examples of 4×4 arrays below left (IV and V)? _____ and _____ .

Will the next example (VI) produce any new results, or will it repeat one already completed?

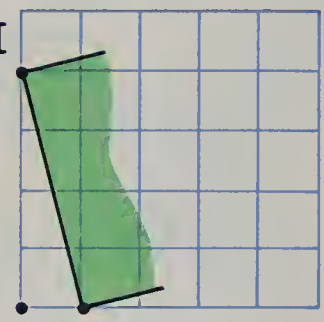
VI



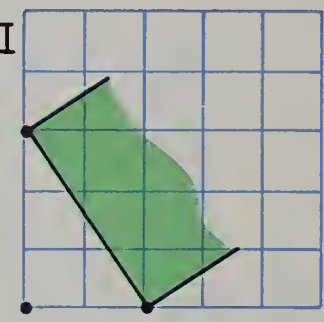
_____

Next, consider two examples based on 5×5 arrays (VII and VIII). Complete the diagram.

VII



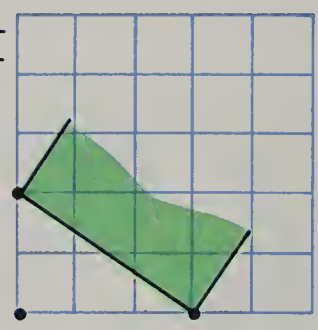
VIII



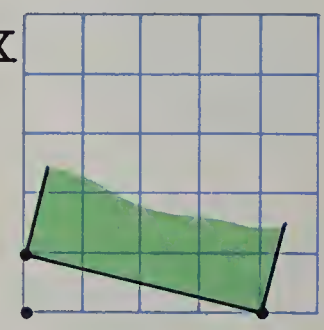
How many tiles are required to cover each of these two squares? _____ and _____ .

Do you think the next two examples (IX and X) will produce any new results? Complete them to make sure.

IX



X



How many tiles are required to cover each of these two squares? _____ and _____ .

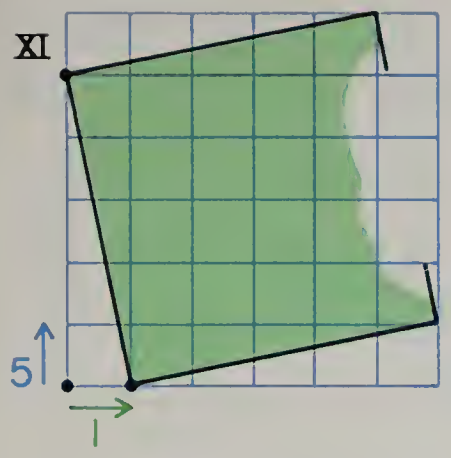
On the chart on the next page, record all of your results.

Complete the diagrams on the next page in the pattern we have used on this page.

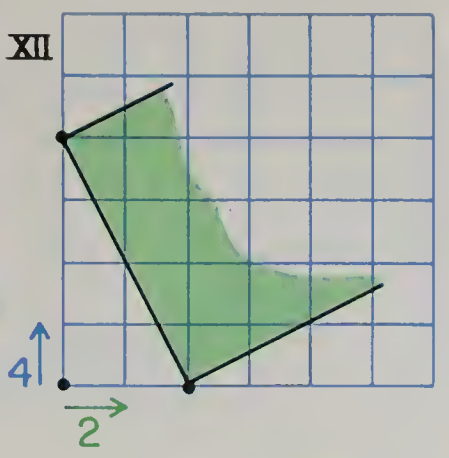
In the chart, what do you notice about the parts tinted blue and green?

Can you design your own experiment to complete the chart?

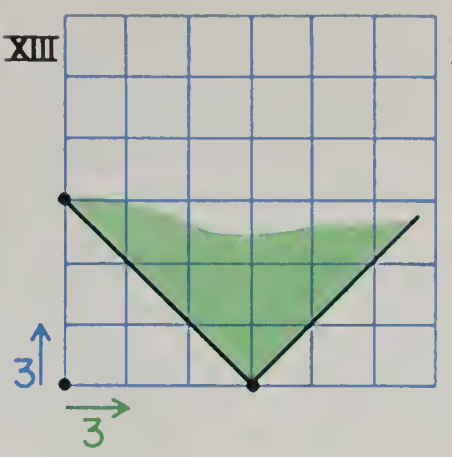
XI



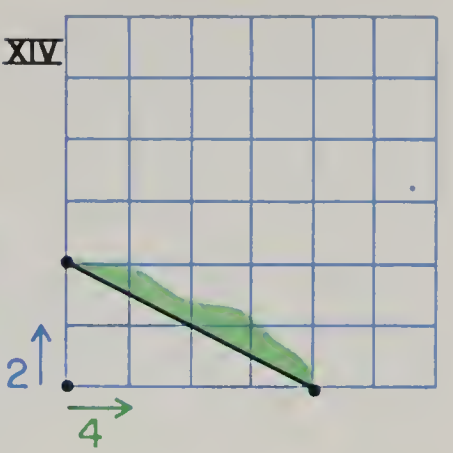
XII



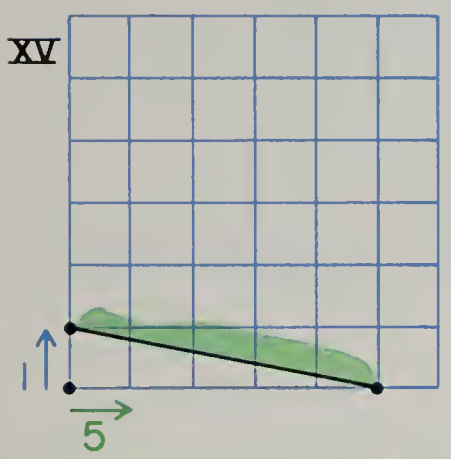
XIII



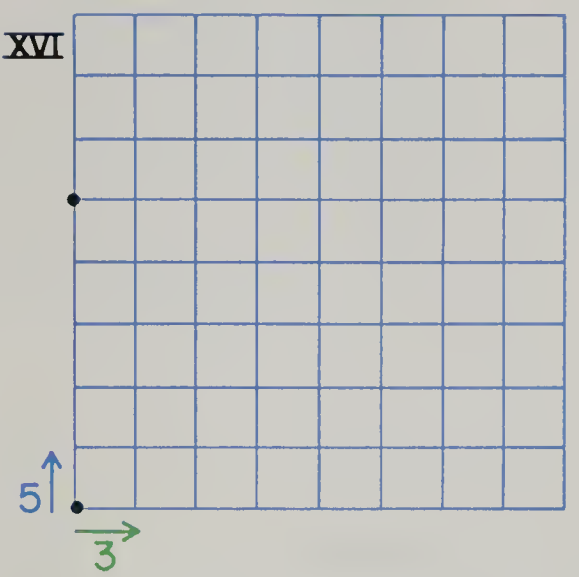
XIV



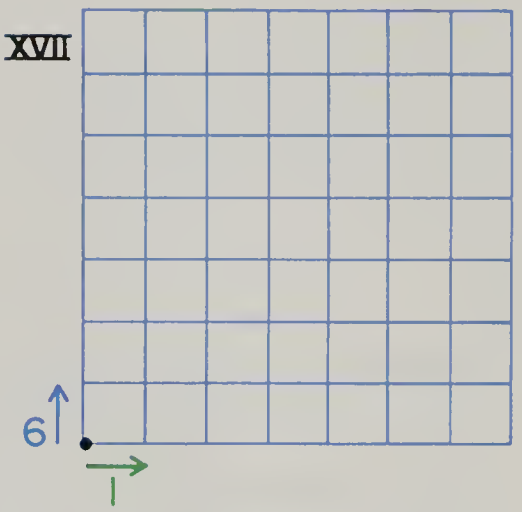
XV



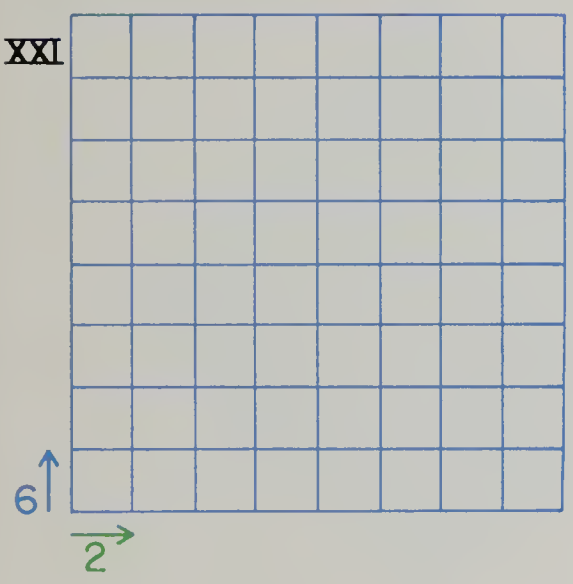
XVI



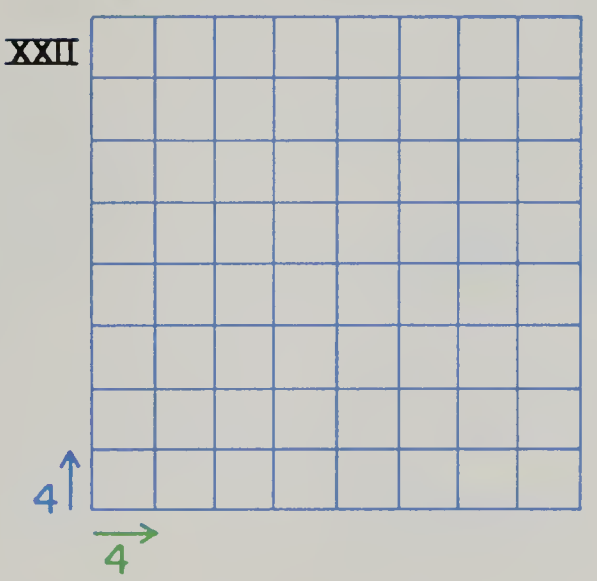
XVII



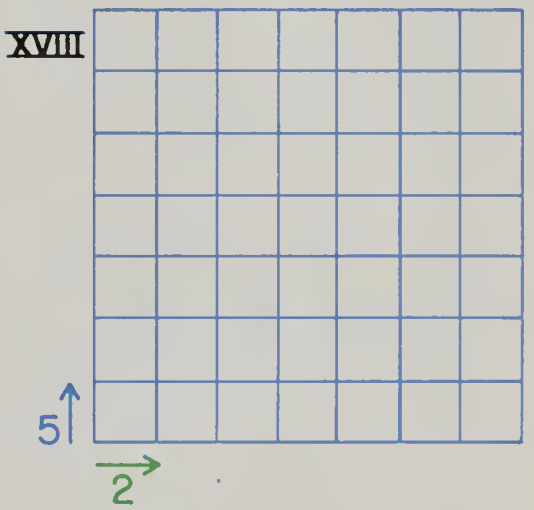
XXI



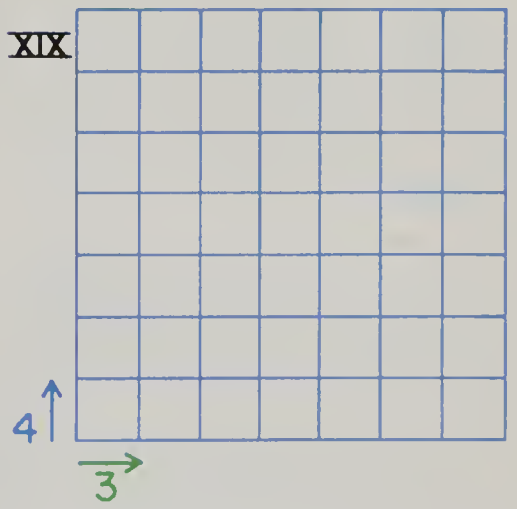
XXII



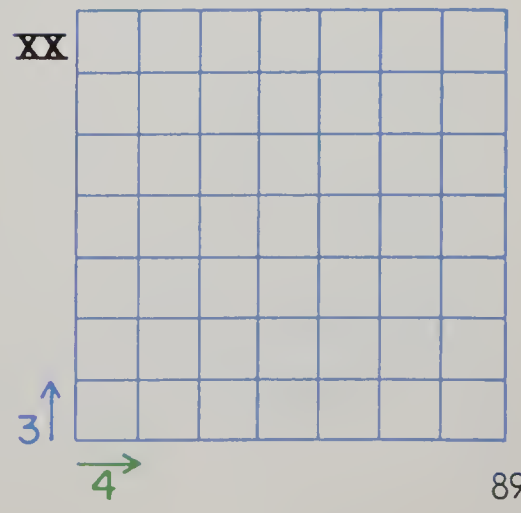
XVIII



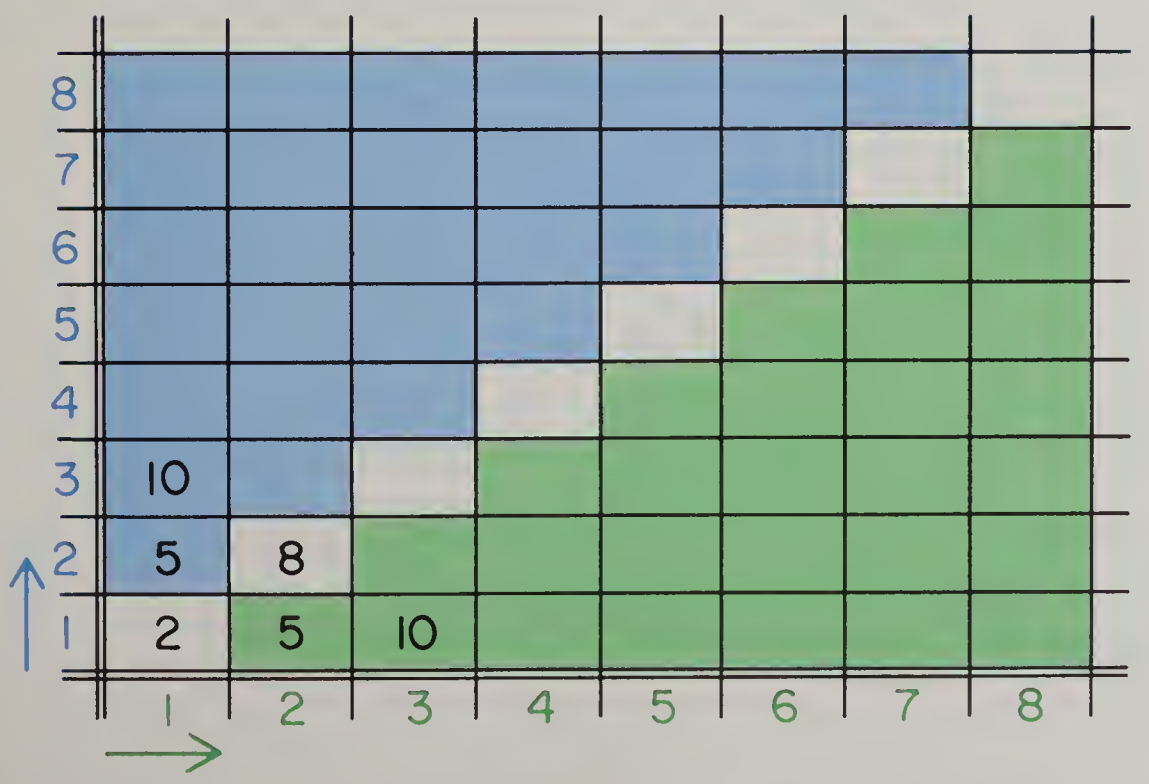
XIX



XX



Record the number of tiles required to cover the square drawn in each example.



A "Pattern" Study!

Let's look carefully at patterns in the table of results for the examples on the previous pages. They are noted in black.

We want to expand the chart by noting patterns—and finding where they lead.

	Y	Z	X	W	V	U	T	S	R
8									
7									
6	37	40							
5	26	29	34						
4	17	20	25	32					
3	10	13	18	25					
2	5	8	13	20					
1	2	5	10	17	26				
	1	2	3	4	5	6	7	8	

a ↑ (2)

b → (1)

Let's begin by studying the numbers noted on the diagonal from the bottom left corner. Numbers in the second and third rows are the difference between the numbers written above them.

R →

2	8	18	32	50	___	___
	6	10	14	18	___	___
		4	4	4	___	___

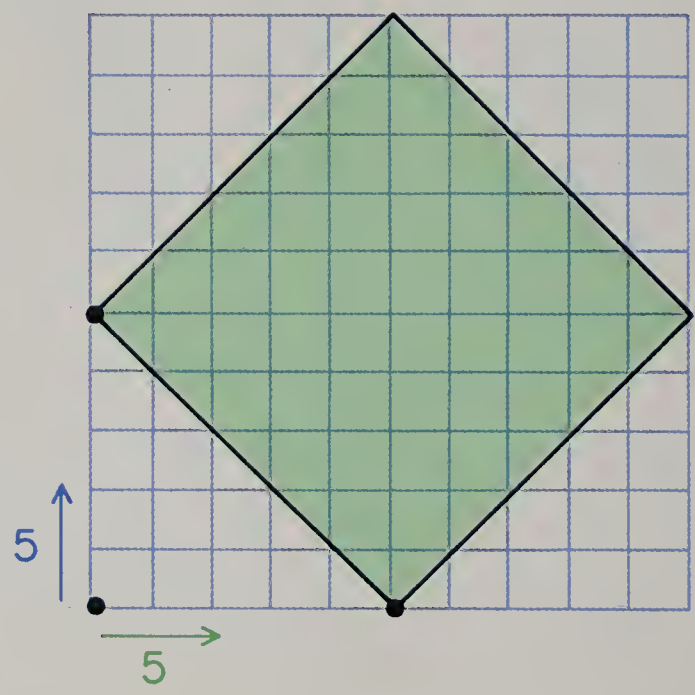
We suspect the bottom row will simply repeat "4." However, we will be more sure if we work out "5 up and 5 to the right." It would help if we found that the result is "50."

You try an example of "6 up and 6 to the right." You will hope that _____ tiles will be required.

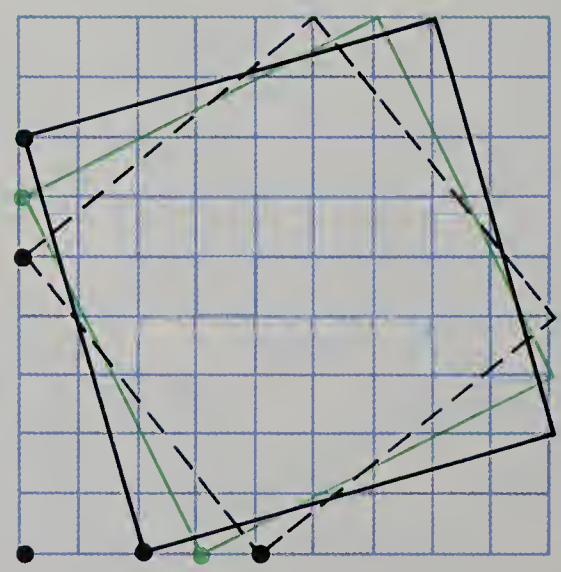
Are your hopes fulfilled? _____

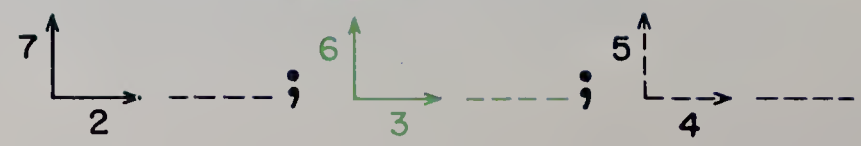
Can you find the result for the uppermost box on diagram R?

Before proceeding, let's investigate to find results for the three boxes shaded blue above. To save space, we can show all three conditions in a single diagram.



It would take ____ tiles to cover the green square.





Now you can fill in all the blue-line blanks from the chart (top row) or by finding the difference of the two numbers written above it (second and third rows):

S

5 13 25 41 ___ ___ ___

8 12 16 ___ ___ ___

4 4 ___ ___ ___

T

10 20 34 ___ ___ ___ ___

10 ___ ___ ___ ___ ___

___ ___ ___ ___ ___

U

17 ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___

V

___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___

W

___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___

Diagonal "X" has no results given. The pattern you see will suggest results for the second and third rows.

X

? ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___

If we know only the first number (on the left) in the top row, we could complete the whole example.

Perhaps a similar study of the left-hand column of results will help. This column is composed of the first number in each of the diagonals we have been studying.

Y

2 5 10 17 ___ ___ ___

3 ___ ___ ___ ___ ___

___ ___ ___ ___ ___

Now you can complete – – → "X."

Did the results in the bottom row surprise you? Let's try the second column from the left in our record.

Z

5 8 13 20 ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

That is monotonously like the previous study. Try the third column from the left.

10 13 18 ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___ ___

___ ___ ___ ___ ___ ___ ___

No longer interesting!

How can you quickly fill in all the spaces below and to the right of the diagonal?

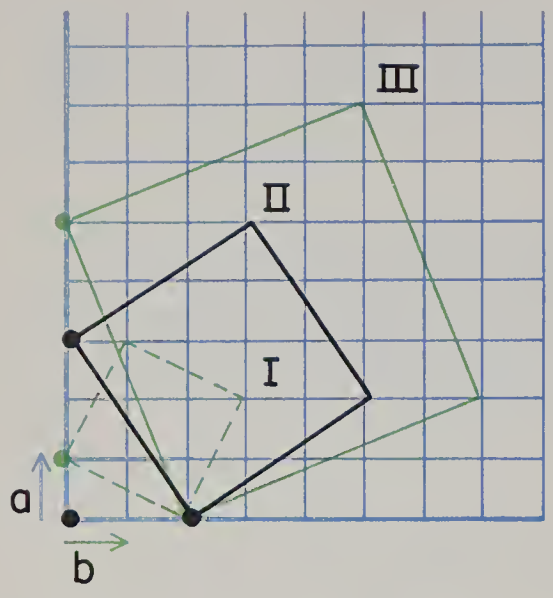
Using the patterns you already know, you can quickly make a chart as large as you want.

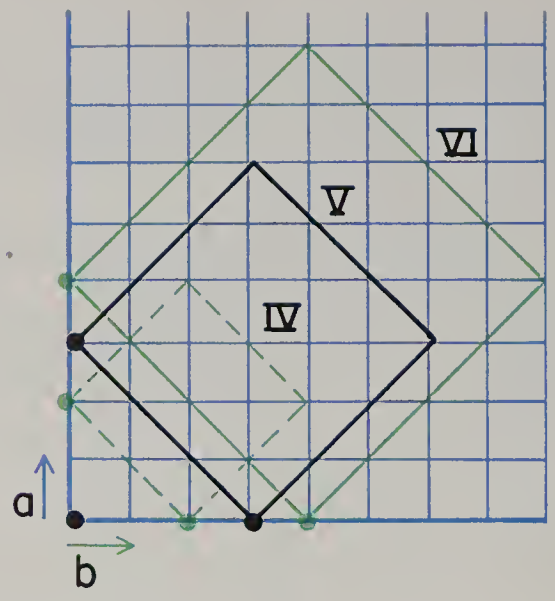
Notice the "over one and up two" diagonal of green boxes. Find more in this series?

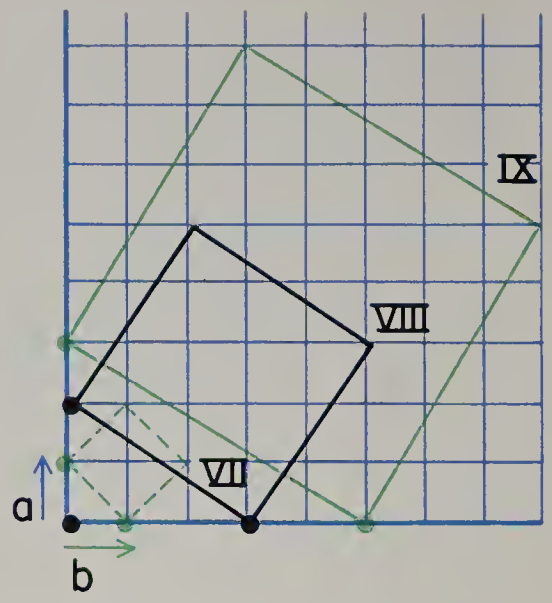
26 45 74 ___ ___ ___ ___

19 ___ ___ ___ ___ ___

___ ___ ___ ___ ___







How many tiles would be required to cover each of the three squares in each of the examples above?

For the last four examples, you may need to draw your own sketches—unless you've discovered a shortcut.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
a ↑___ distance up	1	3	5	2	3					2		3	
b →__ distance to right	2	2		2	3					5	2		3
tiles required to cover squares											13	18	25

A game: "I've got three rules!"

"Call out any two numbers," Mrs. Boswell said. "I will apply each of my rules and write down my results. If you discover the rules, then you know in advance what I am going to write."

Below are some of the results:

Your first number

Your second number

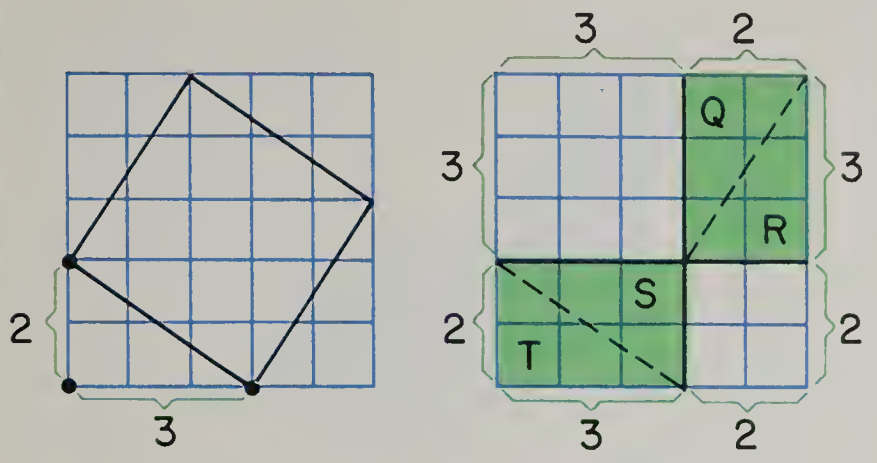
Mrs. Boswell writes

a	b		↓
3	4	→	9
2	7	→	4
5	5	→	25
8	6	→	64
9	1	→	
10	5	→	
a, b		→	
7	4	→	
		→	36
		→	
		→	25

a	b		↓
3	4	→	16
2	7	→	49
5	5	→	25
8	6	→	36
9	1	→	
10	5	→	
a, b		→	
7	4	→	
		→	81
		→	64
		→	

a	b		↓
3	4	→	25
2	7	→	53
5	5	→	50
8	6	→	100
9	1	→	
10	5	→	
a, b		→	
7	4	→	
		→	
		→	113
		→	169

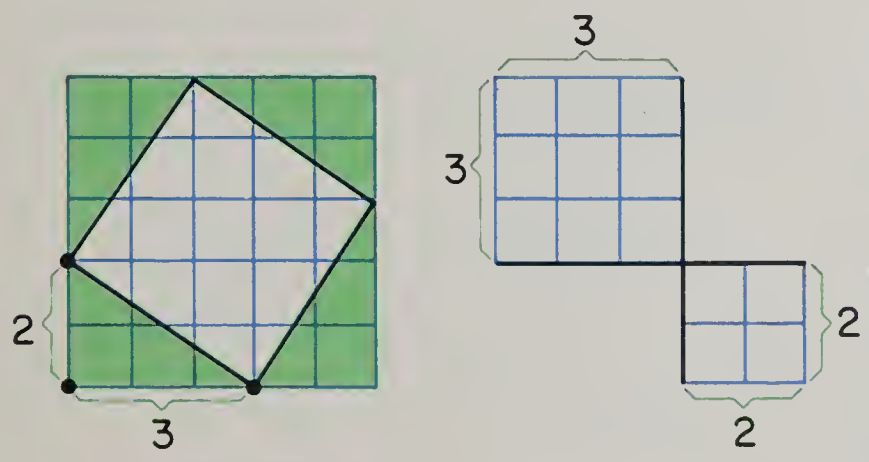
Here is a familiar example: "up 2 and over 3" . . . and a companion grid the same over-all size.



There are 25 tiles in each 5 × 5 array. On the right above, the illustration has certain spaces lettered so we can keep track of them.

Suppose we cut out the pieces tinted green and cut each along the broken line.

We can paste these pieces on top of the diagram on the left. The finished diagram would then appear as below:



From the 5 × 5 grid on the right, we removed 2 · 3 tiles and then 2 · 3 more. Or we removed 12 of the 25 tiles.

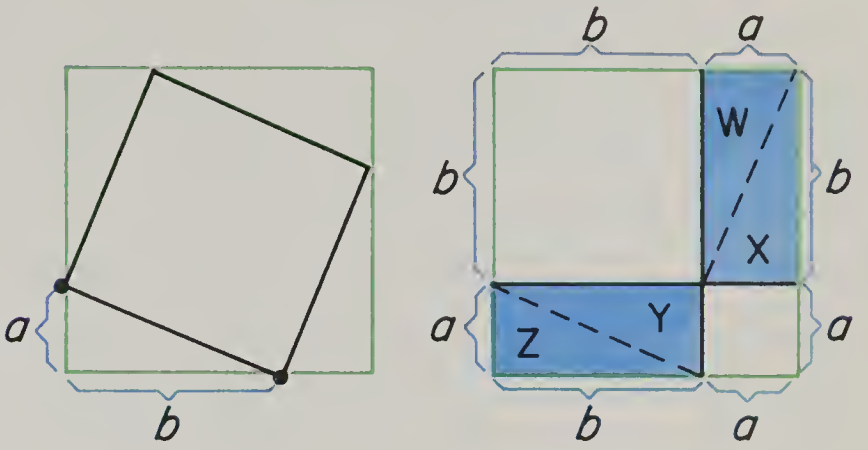
These 12, when cut to fit, will fill the corners outside the tilted square. They cover 12 of the original 25 tiles.

We have left 2 · 2 and 3 · 3 tiles, $4 + 9 = 13$. And, if cut to fit, these 13 tiles certainly ought to cover the tilted square itself.

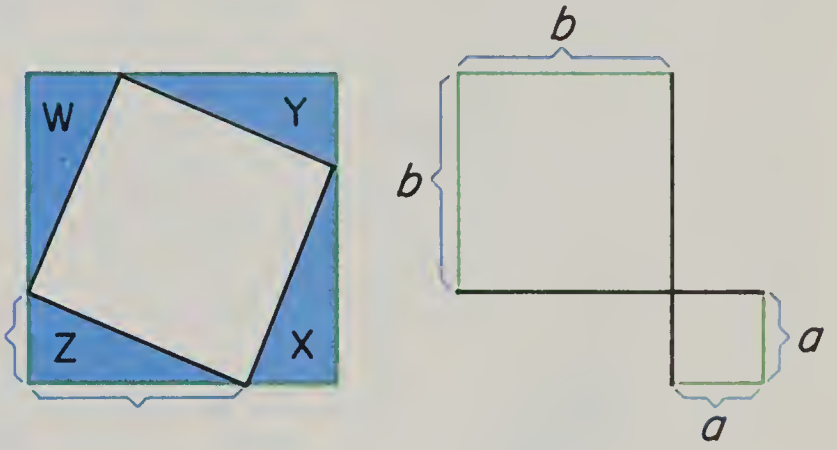
Do you agree?

But there is no longer any charm about that "2 up and 3 over." Think of any two numbers: we will call them "a" and "b" so we can keep track of them—even though we don't know what they are.

Now we draw "a tiles up and b over."



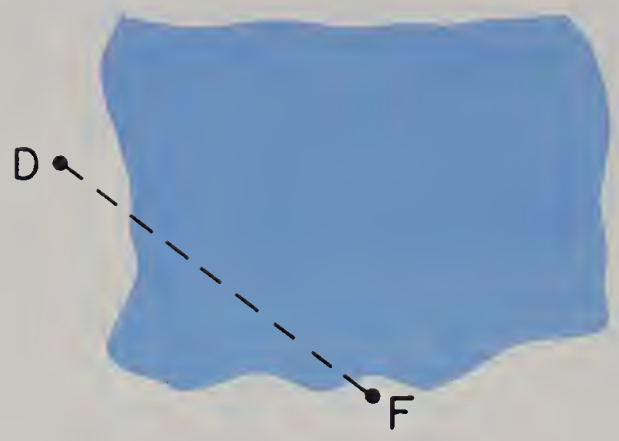
Next we draw in the broken lines and cut out the pieces tinted blue—and cut them on that broken line. After we paste them on the sketch at the left, our results are:



We conclude that $b \cdot b$ tiles and $a \cdot a$ tiles together will be needed to cover the inside of the tilted square in the sketch on the left above. That is $a^2 + b^2$ tiles.

If the two numbers (a and b) that someone thought of are indicated below, you can find the number of tiles needed to cover the square.

	I	II	III	IV	V
a	2	5	7	10	
b	3	5	3	8	4
$a^2 + b^2$					25



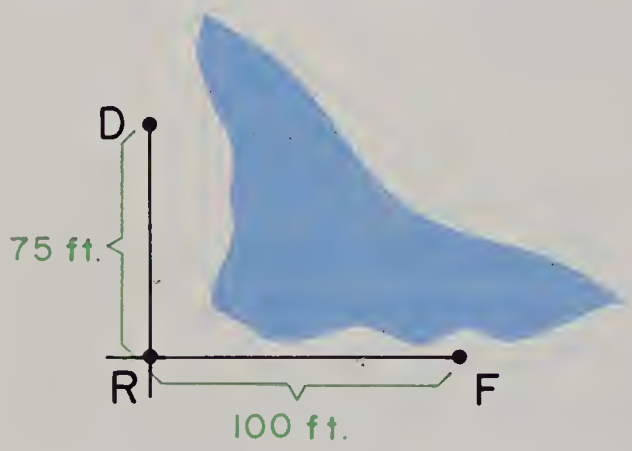
Dan and Fred want to stretch a wire between their homes so they can operate telephones they have made themselves.

How much wire will they need? They can't measure the distance directly, because the wire will cross a lake (as shown in the diagram above) from Dan's house, "D," to Fred's house, "F".

They went about measuring the distance in this way.

They tied a string to the post in front of Dan's house, and another to a post in front of Fred's house.

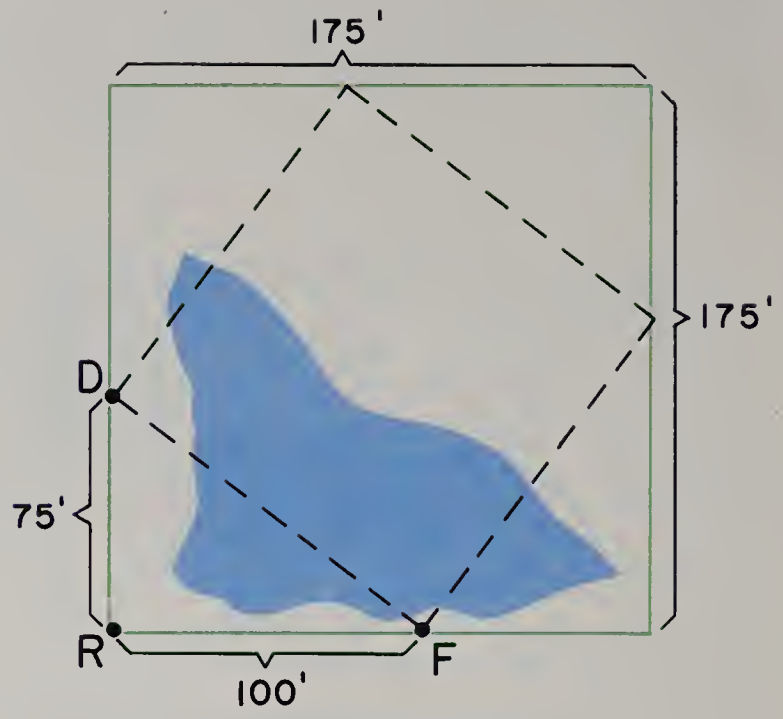
It required considerable experiment but they finally got them to cross at point "R" (below) so that the two strings made a square corner. Like this:



The length of the string from F to R was measured—on dry land. It was 100 ft. long. It took 75 ft. of string to reach from D to R —on dry land.

But they want to know how far it is from D to F—over the water.

So, they made a sketch very much like the sketches we have been studying.



Inside the green square, how many square feet would there be?

175 x 175 = _____

Inside each corner, how many square feet would there be?

$\frac{1}{2}$ x 75 x 100 = _____

In four such corners, how many square feet would there be?

4 x _____ = _____

Then, inside the tilted square indicated by the broken line, how many square feet would there be?

_____ - _____ = _____

What is the length of each side of a square with an area of that many square feet?

_____ = _____ x _____

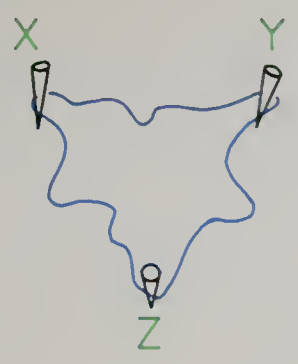
Hint

100 x 100 = _____ 120 x 120 = _____

110 x 110 = _____ 130 x 130 = _____

So the numbers we are looking for must be more than 120 and less than 130.

Marianne made an interesting device using three stakes and three lengths of string. (Stakes are labeled X, Y and Z in following sketch:)



The string from X to Y measured 20 ft. when stretched out straight. The piece between X and Z measured 21 ft., and the piece from Y to Z measured 29 ft.

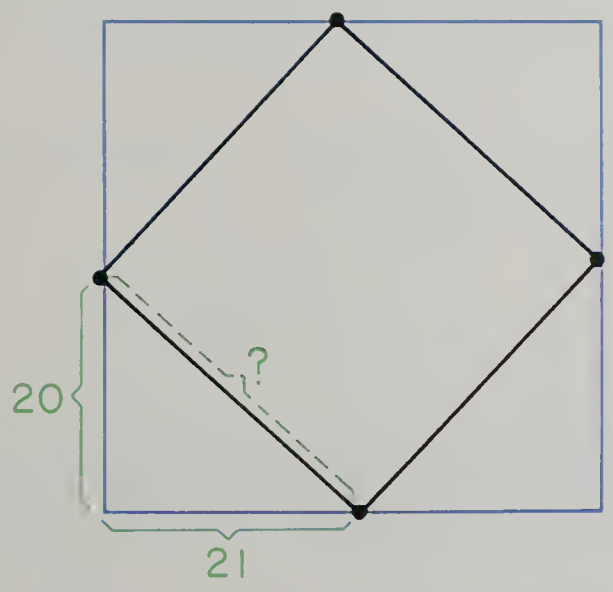
Marianne made the following claim:

"If you drive the three stakes in the ground so that the strings are all stretched out straight, the strings will make a square corner at stake X."

Do you believe she is right?

She explained her device this way:

"One of the examples I tried—like the ones on the preceding pages—gave me these results."



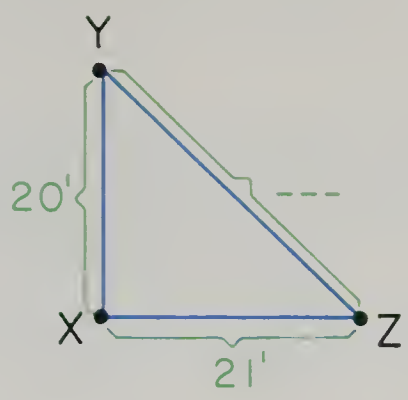
She put some arithmetic on the board and let the class complete it.

20 x 20 = ____ and 21 x 21 = ____

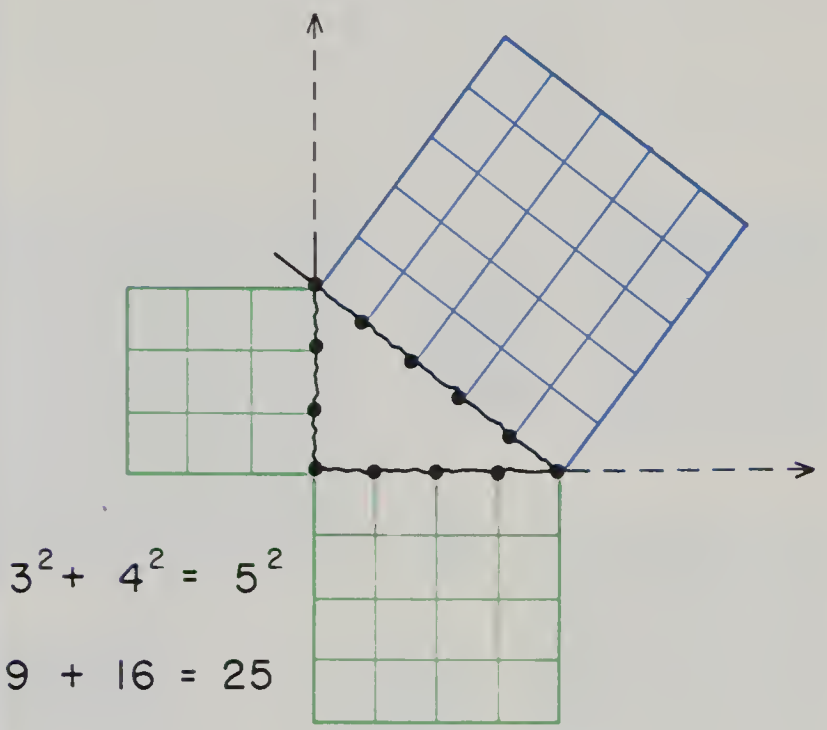
____ + ____ = ____

____ = ___ x ___

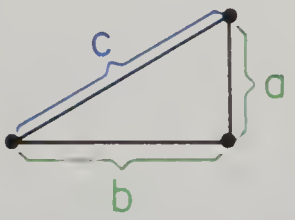
"So," Marianne concluded, "I made my stakes and strings to match."



Early Egyptians used a similar device for working out square corners for their fields. They put knots in a rope the same distance apart. One side was 3 such distances, another 4, and the other was 5. They thought about it this way:



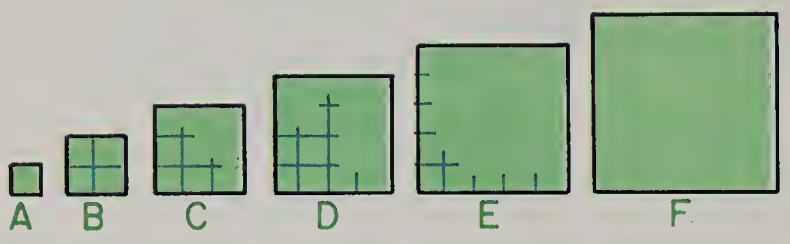
They argued this way: We think of a triangle with sides that are "*a*," "*b*" and "*c*"; and, if $a^2 + b^2 = c^2$, then the triangle has a square corner and is called a "right triangle."



If this is true, is a triangle whose sides are 15", 8" and 17" a "right triangle"? . . . or does $15^2 + 8^2 = 17^2$?

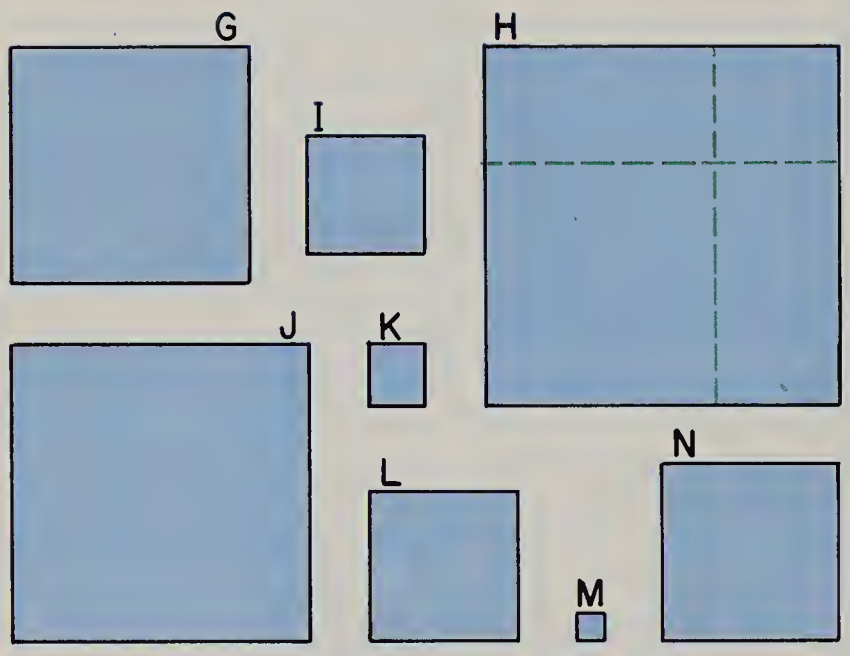
If you know the area of a square, can you find the length of its sides?

In the diagrams below, $\frac{1}{8}''$ represents 1 foot.



	A	B	C	D	E	F
Area in sq. ft.	1	4				
Sides in feet	1					

The following sketches are drawn to scale —1″ stands for 1″. With a ruler, measure the sides. Find the area of each.



Record the length of the sides in numbers of inches and the areas in number of square inches in the above examples.

	G	H	I	J	K	L	M	N
Length of sides	1	$1\frac{1}{2}$						
Area	1							

In these last examples, you measured the length of the sides and then computed the area.

Suppose that you are told that the area of a square is 2 square inches. What is the length of the sides?

This is the same as asking for a number which, when multiplied by itself, is equal to 2; or:

$$S \cdot S = S^2 = 2$$

Is the number we are looking for more or less than 1? _____

Is the number we are looking for more or less than 2? _____

So, we know that it is more than one and less than 2 because:

$$1 \times 1 = 1 \quad \text{and} \quad 2 \times 2 = 4$$

Complete the following:

1.1 x 1.1 = _____ 1.5 x 1.5 = _____

1.2 x 1.2 = _____ 1.6 x 1.6 = _____

1.3 x 1.3 = _____ 1.7 x 1.7 = _____

1.4 x 1.4 = _____ 1.8 x 1.8 = _____

Now we know that the number we are looking for is more than _____ and less than _____, and you might guess that it is closer to the smaller number.

Let's find out more by completing the following computations.

1.41 x 1.41 = _____ 1.43 x 1.43 = _____

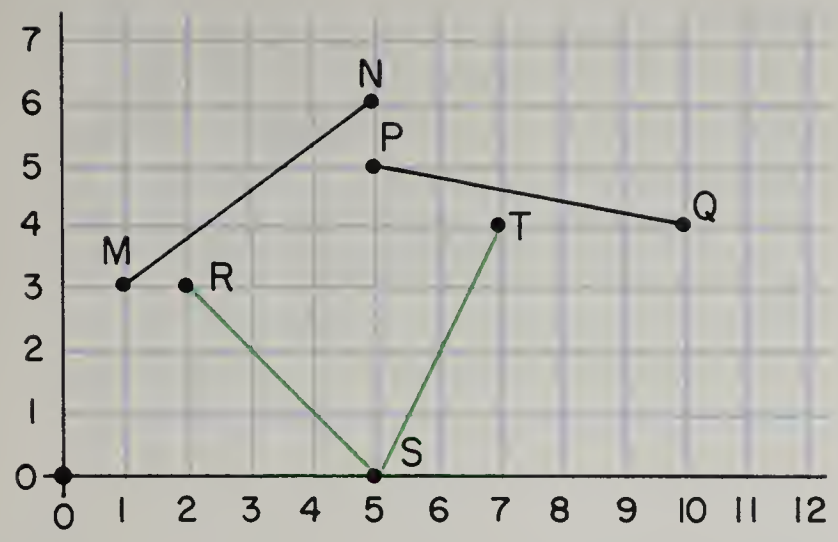
1.42 x 1.42 = _____ 1.44 x 1.44 = _____

Now, we know that the number we are looking for is more than _____ and less than _____.

One number is .0119 too small and the other is .0164 too large. What conclusions can you draw from the following?

1.414 x 1.414 = _____

1.415 x 1.415 = _____



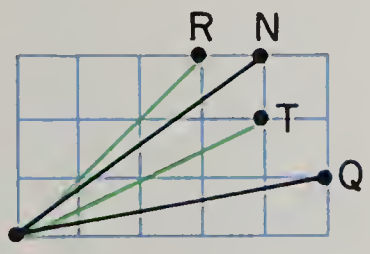
Problem: Find the longest and the shortest line in the sketch above — without measuring.

One approach might be to think of the lines in this way.

$\overline{MN}$ (read "line segment Em En") reaches between opposite corners of a 3×4 array of square units.

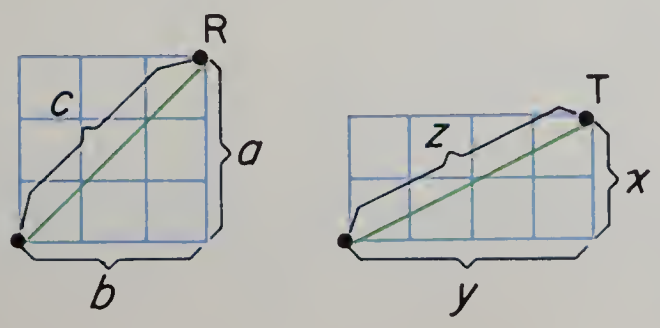
$\overline{PQ}$ reaches between opposite corners of a 1×5 array; $\overline{RS}$ between opposite corners of a 3×3 array; and $\overline{ST}$ between opposite corners of a 2×4 array.

All arrays the same shape and size would be as good as any other, so we can redraw the lines all starting from a single point — labeling only one end to identify the line.



Both black lines seem to be longer than the green lines. Which black line is longer? Which green line is shorter?

Consider the green lines:



Our earlier experiments and study led us to a conclusion we can draw about the situation in these sketches:

$a^2 + b^2 = c^2$ or $9 + 9 = 18 = c^2$
$x^2 + y^2 = z^2$ or $4 + 16 = 20 = z^2$

$4^2 =$ ___ and $5^2 =$ ___

So "c" and "z" are each more than 4 but less than 5. So, follow the procedure used on the previous page.

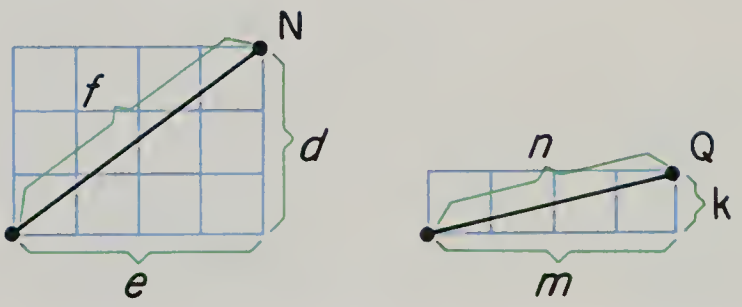
4.1 x 4.1 = ____	4.4 x 4.4 = ____
4.2 x 4.2 = ____	4.5 x 4.5 = ____
4.3 x 4.3 = ____	4.6 x 4.6 = ____

We can see that "c" must be more than ____ but less than ____.

And, we see that "z" must be more than ____ but less than ____.

Therefore, line segment $\overline{RS}$ is _____ (more or less) than line segment $\overline{ST}$.

Next, consider the black lines.



We can conclude that:

$d^2 + e^2 = f^2$ or $9 + 16 = 25$
$k^2 + m^2 = n^2$ or $1 + 25 = 26$

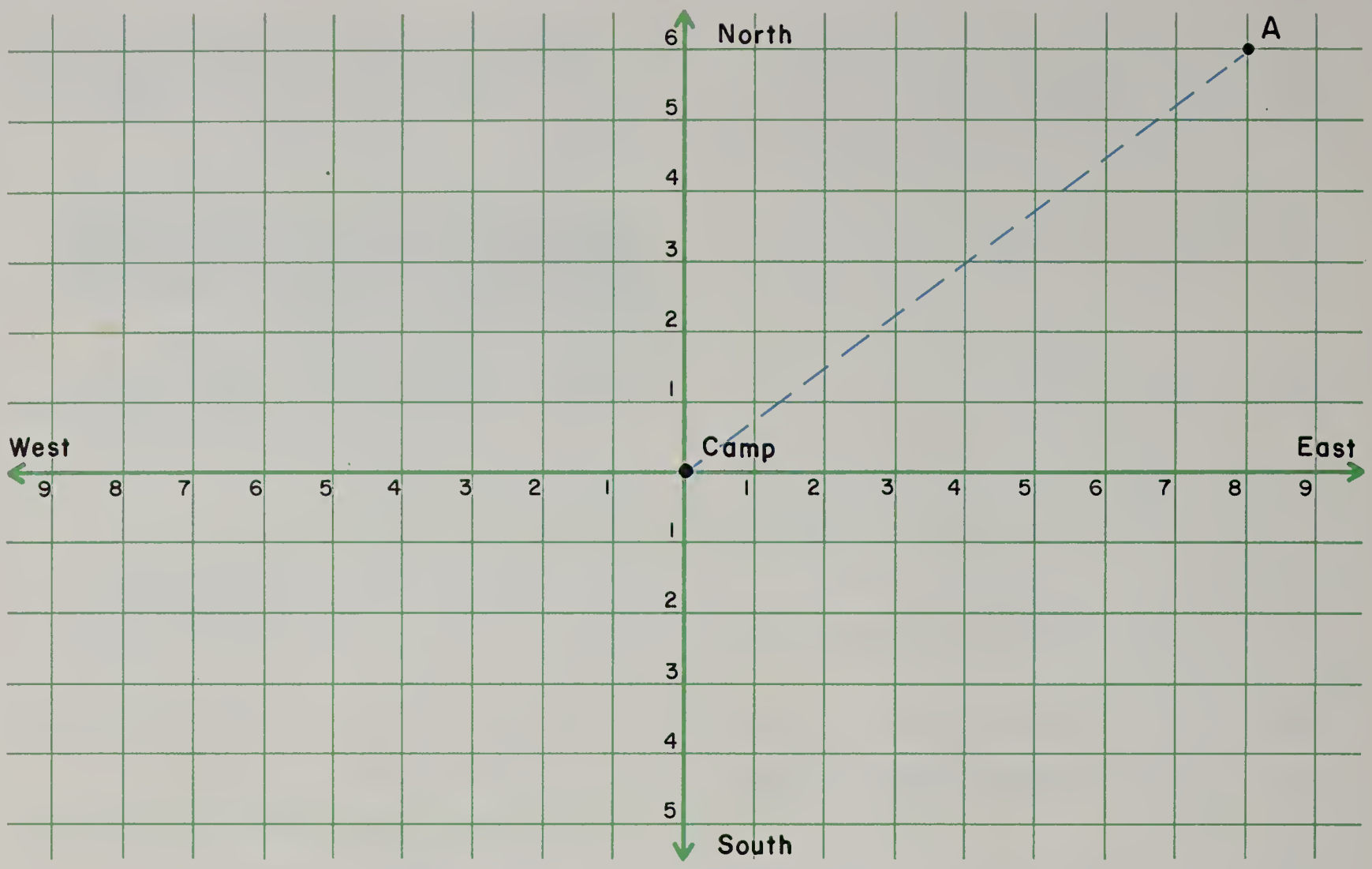
$f^2 = 25$ so $f = 5$.

$n^2 = 26$ so n is _____ (more or less) than 5.

5 x 5 = ____ and 5.1 x 5.1 = ____

We know that "n" is more than ____ and less than _____ .

Conclusion: In our original sketch, line segment ____ is the smallest and line segment ____ is the largest.



A group of 10 prospectors scattered from their camp. After a week's work, each reported his location with respect to the camp.

The radio operator at the camp made a map and marked the position of each man by placing a black dot and labeling it with the first letter of the prospector's name. The operator's notes indicated these positions:

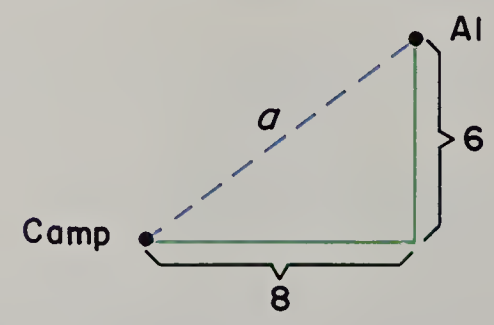
Al: 8 miles east and 6 miles north
Ben: 5 miles west and 2 miles north
Carl: 8 miles west and 6 miles north
Dan: 5 miles east and 5 miles north
Ed: 9 miles west and 5 miles south
Frank: 6 miles east and 5 miles south
Gil: 8 miles west and 2 miles south
Harry: 9 miles east and 2 miles north
Ira: 7 miles east and 3 miles south
Jack: 4 miles west and 5 miles south

Locate each prospector on the map and label the point as the operator did.

The director of the prospecting expedition asked the operator, "How far is Al from camp?"

"Al is 8 miles east and 6 miles north. I'll tell you in a minute the shortest distance by plane."

He drew the following sketch:



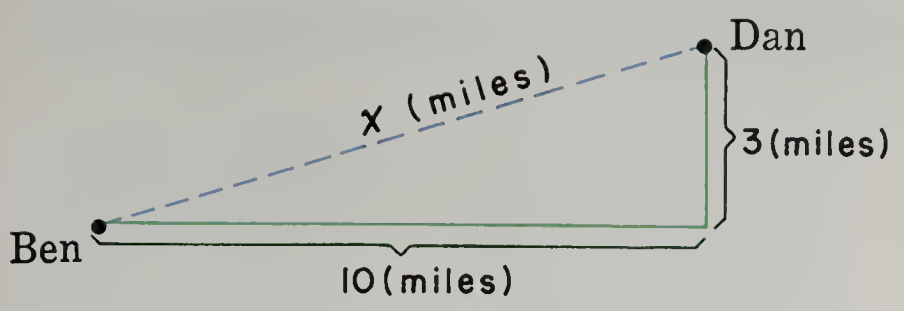
$a^2 = 8^2 + 6^2 = 64 + ____ = ____$

$a = ____$

"Al is _____ miles from camp," the radio operator advised.

"How far is Ben from Dan?"

"Well, Ben is 10 miles west of Dan and 3 miles south of him. I can tell you in a minute the distance between them — as a plane flies."



$x^2 = 3^2 + ___ = ___ + ___ = ____$

Now the operator has a problem. He must look for a number which, when multiplied by itself, or taken as a factor twice, is equal to 109.

10 x 10 = ___ and 11 x 11 = ___

So he knows that the distance is more than _____ miles and less than _____ miles. He decides to try 10.5 . . .

10.5 x 10.5 = _ _ _ _ _ _

The distance is _____ (more or less) than 10.5 miles. The operator makes the following calculations:

10.3 x 10.3 = _ _ _ _ _ 10.4 x 10.4 = _ _ _ _ _

He knows that the distance between Ben and Dan is more than _____ , but less than _____ miles. So, he makes the following calculations:

10.43 x 10.43 = _ _ _ _ _ _ _

10.44 x 10.44 = _ _ _ _ _ _ _

10.45 x 10.45 = _ _ _ _ _ _ _

10.46 x 10.46 = _ _ _ _ _ _ _

"I have found," the operator reported, "that Ben and Dan are more than _____ miles apart, but less than _____ miles apart. Do you want a closer approximation?"

"No," was the answer, "that is close enough — to the nearest hundredth of a mile."

"Please give me a record showing how far each man is from camp. Also, I want to know how far each man is from the other . . . to the nearest hundredth of a mile."

All distances are recorded as numbers of miles. 10.44 + means more than 10.44 miles, but less than 10.45 miles apart.

Al	0										
Ben				10.44+							
Carl											
Dan		10.44+									
Ed											
Frank										10	
Gil											
Harry											
Ira											
Jack						10					
-- to ->	Al	Ben	Carl	Dan	Ed	Frank	Gil	Harry	Ira	Jack	CAMP

"$\sqrt{49} = __$" asks what number taken as a factor twice gives the product 49?

The number is 7, so we can say:

$\sqrt{49} = 7$ $\quad \sqrt{25} = __$ $\quad \sqrt{100} = __$

"$\sqrt{49}$" is read, "the square root of 49," and it is another name for the number 7. We say

$\sqrt{49} = 7$ and $7^2 = 49$

"The square root of 49 is 7," and 7 squared is 49."

If a room is 11 feet wide and 11 feet long, the floor contains ____ square feet.

If the floor of a square room contains 121 square feet, it must be ____ feet by ____ feet.

In the chart below, A stands for the number of square feet in the area of a square floor; and s stands for the number of feet in the length of each side.

	a	b	c	d	e
s	8				
A		100	81	225	144

	f	g	h	i	j
s	21				
A		256	169	400	625

This pattern might be expressed as:

$A = s^2$ or $\sqrt{A} = s$

and it is understood that A refers to the number of square feet in the area of a square, and s refers to the number of feet in the length of each side.

In this sense, we read these expressions as "A equals s squared" and "the square root of A equals s."

But remember that we need to know what A and s refer to, or the expression would not have meaning.

For example: Suppose A referred to the number of square feet in a room that was twice as long as it was wide; and s referred to the number of feet in the shorter side — the width. A chart of some possibilities might look like this:

	k	l	m	n	o
s	8		15		
A		200		288	162

	p	q	r	s	t
s			20		
A	242	392		338	968

This pattern is often expressed as:

$A = 2s^2$ or $\sqrt{\frac{1}{2}A} = s^2$

This certainly is different. You know that the letters do not refer to the same idea in this column and the column to the left.

Complete the following as true statements:

$\sqrt{9} = __$ $\quad \sqrt{16} = __$ $\quad \left(\frac{1}{2}\right)^2 = __$

$4^2 = __$ $\quad \sqrt{\frac{1}{4}} = __$ $\quad 50^2 = __$

$\sqrt{.25} = __$ $\quad 2.5^2 = __$ $\quad \sqrt{625} = __$

$\sqrt{2{,}500} = __$ $\quad \left(\frac{4}{10}\right)^2 = __$ $\quad \sqrt{.09} = __$

$.5^2 = __$ $\quad \sqrt{\frac{1}{16}} = __$ $\quad \sqrt{6\frac{1}{4}} = __$

Betty said that she thought the following statements were true. Do you agree?

(1) $\sqrt{8} = \sqrt{4 \cdot 3} = 2 \times \sqrt{2}$ or $2\sqrt{2}$

(2) $\sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$

(3) $\sqrt{64} = \sqrt{16 \cdot 4} = 4\sqrt{4}$

Let's test the first example. How much is "the square root of 8?" Or, what is the number of feet in the length of a square that has an area of 8 square feet?

We know that $\sqrt{4} = 2$ and $\sqrt{9} = 3$, so $\sqrt{8}$ must be _____ (more or less) than 2 and _____ than 3. We suspect it is closer to 3 than to 2. So, let's make a few trials:

$2.6 \times 2.6 =$ ____ $2.8 \times 2.8 =$ ____

$2.7 \times 2.7 =$ ____ $2.9 \times 2.9 =$ ____

So, $\sqrt{8}$ is more than _____ and less than _____. More trials are in order.

$2.81 \times 2.81 =$ ____ $2.83 \times 2.83 =$ _____

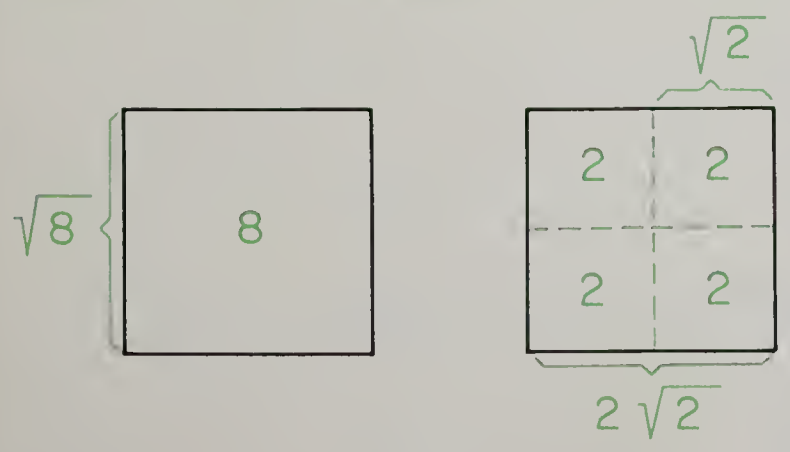
$2.82 \times 2.82 =$ _____

So we know that $\sqrt{8}$ is a little less than _____.

Earlier we found that $\sqrt{2}$ was about 1.414. So, $2\sqrt{2}$ must be about _____.

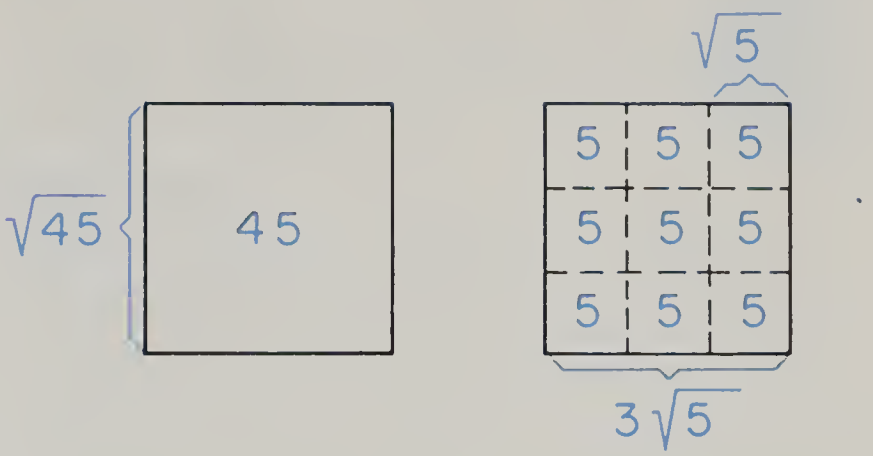
What can you say about Betty's first statement . . . that $\sqrt{8} = 2\sqrt{2}$?

Jim said that he thought Betty was exactly right. He drew two sketches on the board.



Then Jim sat down without saying a word.

Betty knew that Jim knew her secret, so she put another pair of sketches on the board.



Jim came up to the board again. Everyone expected him to draw a sketch to explain Betty's final statement. All Jim wrote was

$\sqrt{64} = 8$ and $4\sqrt{4} = 4 \cdot 2 = 8$

Jim sat down without a word.

Do the sketches and Jim's arithmetic convince you that Betty was right?

Suppose Betty was right. Use the pattern she followed. See if it leads you to any disturbing conclusions.

(Draw sketches if you have a question about the results.)

$\sqrt{12} = \sqrt{\quad \cdot 3} = 2\sqrt{\quad}$

$\sqrt{18} = \sqrt{\quad \cdot \quad} = 3\sqrt{\quad}$

$\sqrt{\quad} = \sqrt{\quad \cdot 2} = 6\sqrt{2}$

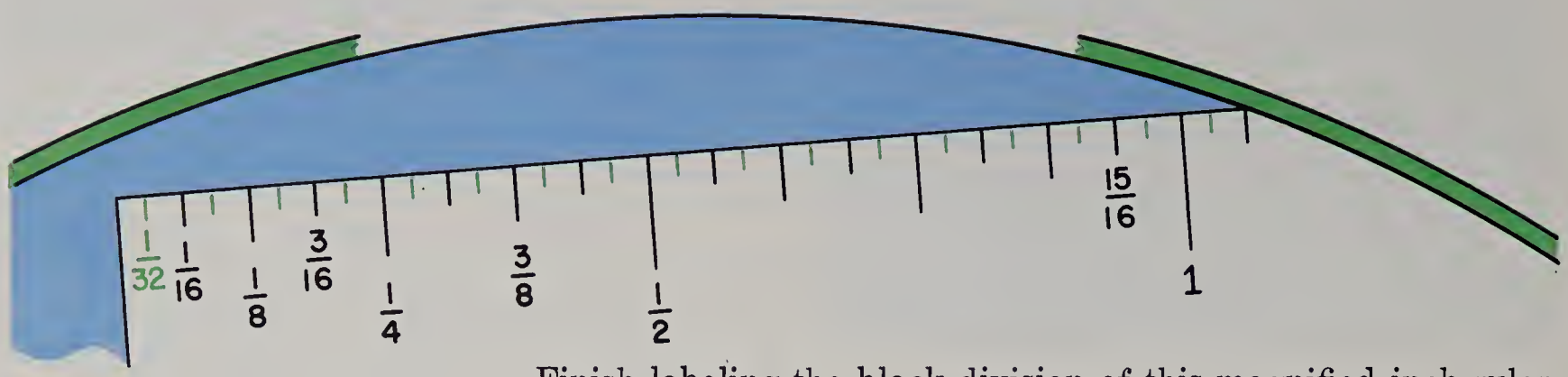
$\sqrt{\quad} = \sqrt{\quad} = 5\sqrt{5}$

$\sqrt{\quad} = \sqrt{\quad} = 4\sqrt{7}$

Indicate the whole number that would make each statement true.

$\sqrt{9} + \sqrt{25} =$ ____ $\sqrt{100} \div \sqrt{25} =$ ____

$\sqrt{49} \cdot \sqrt{121} =$ ____ $\sqrt{3} \cdot \sqrt{3} =$ ____



Finish labeling the black division of this magnified inch ruler.

When engineers work in fractions of an inch, they find their computations are easier if they express parts of an inch in decimal fractions rather than common fractions. Since it takes 3 decimal places to find a decimal fraction that is another name for an eighth, they often express all decimal equivalents in thousandths.

They build a table by doing some arithmetic. From then on, they use the table until they have memorized it. Here is the arithmetic. You fill in the answers.

$\frac{1}{8}$	$8\overline{)1.000}$	$\frac{5}{8}$	$8\overline{)5.000}$
$\frac{1}{4}$	$4\overline{)1.000}$	$\frac{3}{4}$	$4\overline{)3.000}$
$\frac{3}{8}$	$8\overline{)3.000}$	$\frac{7}{8}$	$8\overline{)7.000}$
$\frac{1}{2}$	$2\overline{)1.000}$	$\frac{1}{1}$	$1\overline{)1.000}$

To express sixteenths as decimal equivalents, a fourth decimal place is used.

$\frac{1}{16}$	$16\overline{)1.0000}$	$\frac{9}{16}$	$16\overline{)9.0000}$
$\frac{3}{16}$	$16\overline{)3.0000}$	$\frac{11}{16}$	$16\overline{)11.0000}$
$\frac{5}{16}$	$16\overline{)5.0000}$	$\frac{13}{16}$	$16\overline{)13.0000}$
$\frac{7}{16}$	$16\overline{)7.0000}$	$\frac{15}{16}$	$16\overline{)15.0000}$

An engineer wants to have six holes drilled in a straight line, with centers $1\frac{7}{16}$ inches apart. What is the distance between the centers of the two holes that are furthest apart?

Answer: _____

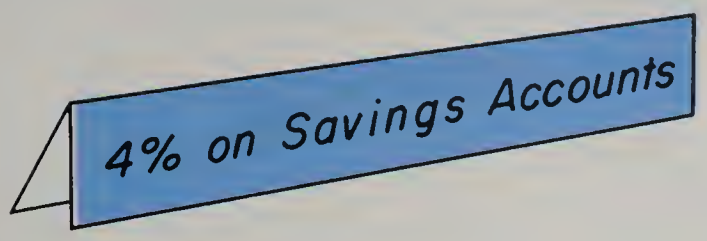
When the engineer measured the finished piece, he found that the end holes were .007 of an inch further apart than he had requested. They were _____ apart.

How might an engineer express the following as decimal fractions?

$1\frac{1}{16}''$ →	_____	$7\frac{11}{16}''$ →	_____
$12\frac{3}{8}''$ →	_____	$4\frac{1}{4}''$ →	_____
$3\frac{1}{5}''$ →	_____	$17\frac{3}{5}''$ →	_____
$\frac{1}{20}''$ →	_____	$\frac{7}{25}''$ →	_____
$\frac{17}{50}''$ →	_____	$1\frac{3}{20}''$ →	_____

Engineers often need to work in thirty-seconds and sixty-fourths of an inch. Change the following common fractions to decimal fractions:

$\frac{1}{32}$ =	_____	$\frac{1}{64}$ =	_____
$\frac{6}{32}$ =	_____	$\frac{7}{64}$ =	_____
$\frac{11}{32}$ =	_____	$\frac{40}{64}$ =	_____
$\frac{28}{32}$ =	_____	$\frac{19}{64}$ =	_____



This sign in a bank certainly wouldn't mean anything to a visitor from Mars. Think of all the ideas you need to understand so that the sign makes sense.

4% . . . means 4 per hundred or 4 out of a hundred and it is another name for the fraction $\frac{4}{100}$ or $\frac{2}{50}$ or $\frac{1}{25}$. It can be written in decimal fraction form as .04.

Savings account . . . refers to a record kept of the money a person loans a bank. Because the owners of the bank can put the money to use, they are willing to pay rent on the money. This rent is called

Interest . . . The person who planned the sign assumed that you would understand that "4%" means "4% interest." If you put $85 in the bank and receive 4% interest, you would receive .04 × $85 as interest, which is $_____. But how long does the bank think it can use that $85 by paying that amount of rent or interest? Again, it is assumed that, unless otherwise indicated, the indicated amount of interest pays rent for

One year . . . If you keep the money in the bank for two years, you will be paid 4% of the amount as interest for each year. If you leave it only one-half year, you will receive $\frac{1}{2}$ of the amount you would have received after one year.

These are some of the ideas you need to know before that sign means anything to you.

"Principal" is a word that refers to the amount of money for which rent or interest is to be paid.

"Rate" is "per cent per year" such as "4% per year" in this case.

"Interest" is the amount paid for the use of the "principal."

A part of an interest chart might look something like the following. Complete the open spaces.

Principal	Rate	Interest	Rate	Interest
$ 1	4%	$ .04	3%	$ .03
2	4%	.08	3%	
3	4%		3%	
4	4%		3%	
5	4%		3%	
6	4%		3%	
7	4%		3%	
8	4%		3%	
9	4%		3%	
10	4%		3%	
20	4%		3%	
30	4%		3%	
40	4%		3%	
50	4%		3%	

To find the interest on $47 at a rate of 3%, look up the amount of interest on $40 and on $7; then add those amounts.

3% interest on $47 is _____

4% interest on $34 is _____

3% interest on $59 is _____

Numbers Have Many Names!

Some of the expressions scattered about the top section of this page are other names for the numbers at the head of the columns in the chart below. Cross out the name in the top section, and write it in the proper column in the chart.

$12 \div \frac{1}{2}$ $\sqrt{9} \cdot \sqrt{9}$ 50% of 1 $100 \times .1$ 100% XXIV IX $2 \times 25\%$ $995 \div 995$ $\sqrt{100}$ 10% of 100 $1\% \times 1{,}000$ $5\sqrt{4}$ $.3 \div 40\%$ $\frac{1}{2}\sqrt{400}$ X 75% $194\overline{)97}$ $\sqrt{576}$ I $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}$ XIX 97×9.7 $\sqrt{90}$ $18-19+2$ $\frac{1}{4}\sqrt{9}$ $\frac{84}{7}$ $40\% \div \frac{2}{5}$ $50\% \div \frac{1}{2}$ $\frac{3}{9} \cdot \frac{150}{5}$ $5 \div 50\%$ 75% of 1 $\frac{15}{16}-\frac{7}{16}$ $\frac{14}{7} \times \frac{50}{10}$ $\sqrt{49+2}$ $.1 \div .1$ 4^2+8 $\frac{1}{2} \times 50\%$ $\sqrt{1}$ $4\sqrt{36}$ $\sqrt{18}$ $\sqrt{81}$ $6 \div \frac{1}{4}$ $97\overline{)35{,}876}$ $\frac{63}{84}$ 900% $124\overline{)93}$ $2^2 \times \sqrt{36}$ $\sqrt{1{,}000}$ $.7\overline{)16.8}$ $7\frac{1}{8}-6\frac{1}{8}$ $7\overline{).7}$ 2 x .375 $\frac{1}{4} \div \frac{1}{2}$ $408 \div 17$ $6 \div \frac{2}{3}$ 25% of 2 1,000% $4 \times 12\frac{1}{2}\%$ $18\% \times 2$ $\frac{3}{4} \div 1\frac{1}{2}$ $2\sqrt{25}$ $3 \div \frac{1}{3}$ $200\% \times 12$ $\frac{3}{5}+\frac{3}{20}$ $\frac{\sqrt{21}}{\sqrt{21}}$ $4\% \times 225$ $1.1-\frac{3}{30}$ 30 x .025 $30\% \times 30$ $\sqrt{\frac{9}{16}}$ $3 \div \sqrt{9}$ $1-(10 \times 21)$ $111-87$ 10% of 5 $8 \times .0625$ $100\overline{)900}$ $16 \times .03125$ $7645-7636$ $\frac{5}{21}+\frac{11}{42}$ $2 \times 3 \times 4$ $\frac{351}{39}$ $-5+{}^{+}15$

Other names for . . . . .

10	$\frac{3}{4}$	1	24	9	$\frac{1}{2}$

reverse digit pairs	sum	difference
21 , 12	33	9
53 , 35	88	
91 , 19		
10 , 01		
84 , 48		
62 , 26		
96 , 69		
83 , 38		
73 , 37		
81 , 18		
20 , 02		
82 , 28		
92 , 29		
31 , 13		
,	55	9
,	77	27
,	66	18
,	99	27
72 ,		
90 ,		
, 07		
, 4		
65 ,		
75 ,		
85 ,		
95 ,		

Pairs of Numbers with Reverse Digits

In each column below, list pairs of reverse digit numbers in which the difference between the members of the pairs is indicated at the top of the column.

9	18	27	36	45
32 , 23	53 , 35	96 , 69	,	,
87 , 78	,	,	,	,
,	,	,	,	,
,	,	,	,	,
,	,	,	,	,
,	,	,	,	,

In each column below, list reverse digit pairs in which the sum of each pair is listed at top of the column.

77	88	99	121	132
70 , 07	,	,	,	,
61 , 16	,	,	,	,
,	,	,	,	,
,	,	,	,	,

Add

72	31	96
27	13	69

Subtract

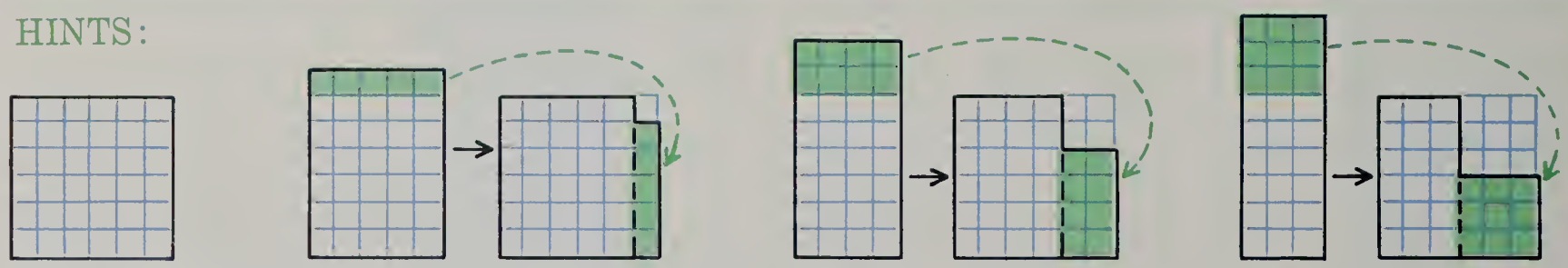
76	93	75
67	39	57
84	71	62
48	17	26

reverse digit pairs	sum	difference
73 , 37	____ x 11	____ x 9
61 , 16	____ x 11	____ x 9
53 , 35	____ x 11	____ x 9
92 , 29	____ x 11	____ x 9
54 , 45	____ x 11	____ x 9
,	8 x 11	4 x 9
,	9 x 11	7 x 9
,	4 x 11	4 x 9

Find the missing products indicated below. In the green-tinted spaces, rewrite these products by completing the statement. If you find shortcuts, use them.

20 X20 400	21 X19	22 X18	23 X17	24 X16	25 X15	26 X14
400 − 0	400 −	400 −	400 −	400 −	400 −	400 −
30 X30	31 X29	28 X32	27 X33	34 X26	35 X25	24 X36
900 −	900 −	900 −	900 −	900 −	900 −	900 −
50 X50	49 X51	52 X48	47 X53	54 X46	45 X55	56 X44
2,500 −	2,500 −	2,500 −	−	−	−	−
80 X80	79 X81	78 X82	77 X83	76 X84	75 X85	74 X86
6,400 −	6,400 −	6,400 −	−	−	−	−
60 X60	41 X39	72 X68	87 X93	64 X56	75 X65	96 X84
3,600 −	1,600 −	− 4	− 9	−16	−25	−36
90 X90	89 X91	38 X42	67 X73	76 X84	35 X45	66 X54
8,100 −	− 1	1,600 −	−	−	−	−

HINTS:

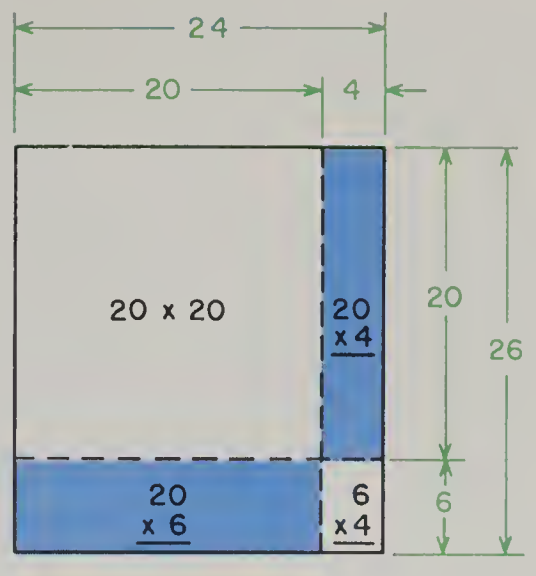


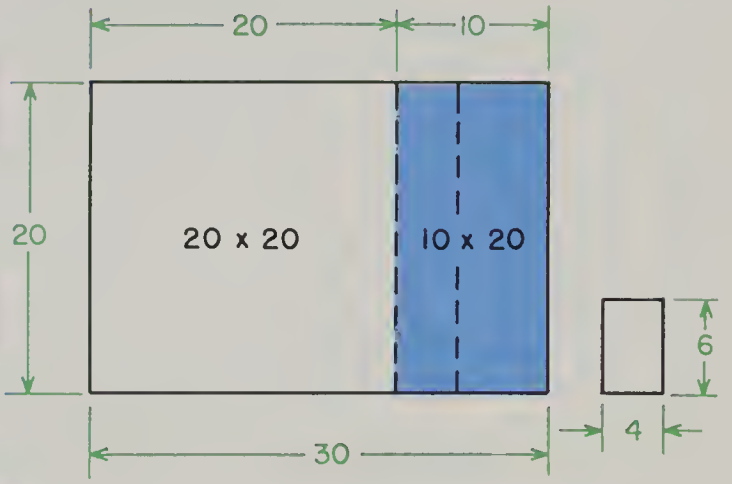
(6+0) (6−0)=6·6− ____	(11+0) (11−0)=11·11− ____	(50+0)(50−0)=50·50− ____
(6+1) (6−1)=6·6− ____	(11+1) (11−1)=11·11− ____	(50+1)(50−1)=50·50− ____
(6+2) (6−2)=6·6− ____	(11+2) (11−2)=11·11− ____	(50+2)(50−2)=50·50− ____
(6+3) (6−3)=6·6− ____	(11+3) (11−3)=11·11− ____	(50+3)(50−3)=50·50− ____

Find the products in each example. Use a piece of scratch paper and on this page record only the results. If you see a reliable pattern, you may not have to work out all the examples. Try to find a shortcut — and use it.

05 x 05 = 0025	06 x 04 = 24	03 x 07 =	08 x 02 =	09 x 01 =
15 x 15 = 0225	14 x 16 =	17 x 13 =	18 x 12 =	11 x 19 =
25 x 25 = 0625	24 x 26 =	27 x 23 =	22 x 28 =	21 x 29 =
35 x 35 =	36 x 34 =	33 x 37 =	32 x 38 =	39 x 31 =
45 x 45 =	46 x 44 =	47 x 43 =	48 x 42 =	49 x 41 =
55 x 55 =	54 x 56 =	53 x 57 =	52 x 58 =	51 x 59 =
65 x 65 =	64 x 66 =	63 x 67 =	62 x 68 =	61 x 69 =
75 x 75 =	76 x 74 =	77 x 73 =	78 x 72 =	79 x 71 =
85 x 85 =	86 x 84 =	87 x 83 =	88 x 82 =	89 x 81 =
95 x 95 =	94 x 96 =	93 x 97 =	92 x 98 =	91 x 99 =
105 x 105 =	106 x 104 =	107 x 103 =	108 x 102 =	109 x 101 =

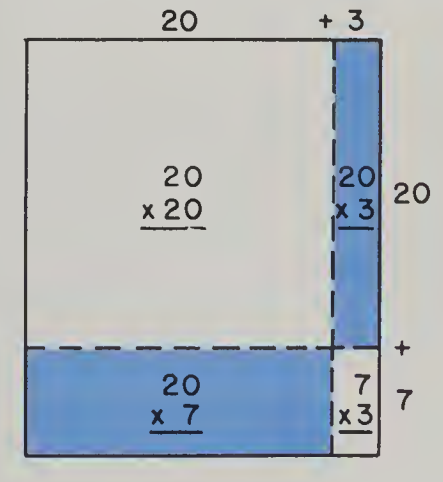
HINTS:

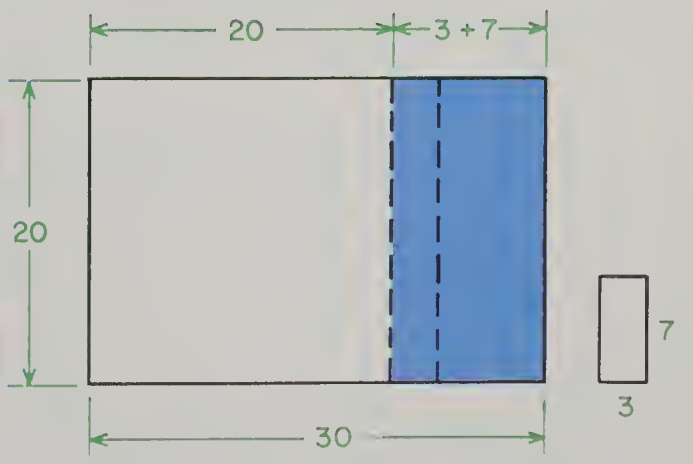




24 x 26 = 20 x 30 + 4 x 6

= 600 + 24 = _____





27 x 23 = 20 x 30 + 3 x 7

= 600 + 21 = _____

Imagine that the telephone rings and you answer: "Hello."

"This is X-Nighyun in a spaceship near your earth's moon. I'm from Mars. Alec Marson, my friend, once visited earth. He taught me to speak your language and told me to phone you if I ever got near your planet. He said you would tell me about arithmetic. Do you know about arithmetic?"

What would you answer?

"Why, yes, of course I know what arithmetic is."

"Well, what is it?"

"It has to do with numbers, and . . ."

"What are numbers?"

Now you begin to realize that you have a big job on your hands.

"Do you have hands?"

"Yes, I am using a hand to guide my spaceship, another hand to hold the sandwich I'm eating, another hand to hold the book I'm reading, and my other hand is in my pocket . . . Sure I have hands."

"That's four hands and four is a number."

"Do you mean that numbers are hands and hands are numbers?"

"No! No! No! No! . . . You see, I said 'no' four times." You tap the telephone with a pencil — tap, tap, tap, tap. "That's four noises. Boom! Boom! Boom! Boom! . . . That's '4' booms . . . a boom for each of your hands . . . No, No, No, No . . . tap, tap, tap, tap, Boom, Boom, Boom, Boom . . . hand, hand, hand, hand . . . four 'hands,' four 'booms', four 'taps', four 'no's' . . ."

"Ah, hah! . . . and you gave me four examples. If there's enough for each hand and no more, that's four. Now I can tell you that I have four fingers on each hand."

"Another number is 'one' . . . you have one hand in your pocket; our earth has one moon; our planet revolves around a single sun; you are one Martian."

"Herchoo," X-Nighyun explodes over the phone, "and I suppose that is one sneeze."

"Yes!"

"One 'yes.' "

This could go on for a long time, so you get on with your job.

"Tie a string around one finger of the hand you have in your pocket."

"One string around one finger."

"Now tie another string around another finger on that hand."

"Completed!"

"Two is another number . . . two pieces of string, two fingers with string on them . . . Boom, Boom . . ."

"Say, then on that same hand I now have two fingers that do not have strings on them. Right?"

"Right. Three is another number. You have three hands that don't have any strings on any of your fingers . . . hand, hand, hand."

"Three, four, one, two — that's four numbers. Is that all the numbers there are . . . as many numbers as I have fingers on one hand? That's easy!"

"No, there's more . . . lots more. I have 'five' fingers on each hand — that's one more than four . . . you see, one plus four is the same number as five."

"Plus? . . . What's that all about?"

"That's a word we use to indicate addition."

"What is addition?"

About this time, you wish no one had given X-Nighyun your number.

"I'm tired. Can you call me back tomorrow for another lesson?"

"I certainly will!"

Wearily you say "Goodbye" and hang up. Questions buzz in your head . . . What are numbers? . . . What is addition? . . . How will you ever explain fractions? . . . Maybe you had better not mention multiplication.

One thing you know — you have a hard job ahead of you.

A Plan of Attack!

There are some difficult tasks to be done. You have learned so much arithmetic over the years that you wonder how it can all be explained over the telephone even if you had a year.

It sure would help if you could explain big chunks of arithmetic to X-Nighyun at a time. Are there some patterns that would help you do this?

Usually, the biggest ideas are so simple you are apt to overlook them. Suppose we tackle addition in an unusual way. We will talk to X-Nighyun this way:

"Whenever you think of any two numbers — alike or different — and also think of addition, a particular number is brought to mind.

"When we think of 2 and 3 and of addition, we connect those ideas with 5 and we say:

"Two plus three is five.

"First, I will tell you the numbers I am thinking of along with addition, and then I will tell you the particular number that this brought to mind."

1 , 1 → 2	2 , 1 → _	3 , 1 → _
1 , 2 → 3	2 , 2 → _	3 , 2 → _
1 , 3 → _	2 , 3 → _	3 , 3 → _

We can name this pattern:

$$a + b = c$$

(We are, of course, skipping that part of the story in which X-Nighyun learns to count beyond 5.)

If X-Nighyun is a quick thinker, he will notice something right away.

"That's interesting. I see a pattern that will help me memorize all these and more combinations. The order in which you think of the two numbers doesn't seem to matter:

1 , 2 → 3	2 , 3 → 5	1 , 3 → 4
2 , 1 → 3	3 , 2 → 5	3 , 1 → 4"

"That is right. We can give that pattern a simple name:

$$a + b = b + a$$

and mathematicians call this the Commutative Law of Addition."

"Fine . . . and I think I've got a little idea of addition. I put up one finger. Then I put up two more. Now I have three fingers up — 1, 2 → 3 — and all the other combinations work out the same way."

You would agree with X-Nighyun and then explain that we say that:

"One plus two equals three."

Also, we refer to the pair of numbers we think of along with the idea of addition as "addends." The particular number named in this way is called the "sum" of the addends.

Then X-Nighyun might pose a problem.

"Suppose I put up two fingers on one hand, three on another, and four on another. Which pair do I add first?"

"Try it in any order you like."

You need to explain that:

$2 + (3 + 4) = 2 + __ = __$
$(2 + 3) + 4 = __ + __ = __$
$6 + (4 + 7) = __ + __ = __$
$(6 + 4) + 7 = __ + __ = __$ etc.

or

$2 + (3 + 4) = (2 + 3) + __$
$6 + (4 + 7) = (6 + 4) + __$

This pattern can be explained clearly by using letters to indicate any three numbers you may think of:

$$a + (b + c) = (a + b) + c$$

Mathematicians call this the "Associative Law" of addition.

At this point, X-Nighyun might tell you:

"Those two laws — the Commutative and Associative Laws of Addition — together tell me that it doesn't make any difference in what order we add numbers."

"That's right. You might call them the 'in any order' laws for addition."

"We also have similar 'in any order' rules for multiplication."

"Not so fast. What is multiplication?"

"Whenever we think of any two numbers and also think of multiplication, a particular number is brought to mind. If I think of 2 and 3 and also of multiplication, I have named 6. Here are more examples:

$1, 1 \rightarrow 1$	$2, 1 \rightarrow __$	$3, 1 \rightarrow __$
$1, 2 \rightarrow 2$	$2, 2 \rightarrow __$	$3, 2 \rightarrow __$
$1, 3 \rightarrow __$	$2, 3 \rightarrow __$	$3, 3 \rightarrow __$

"We refer to the pair of numbers we think of along with the idea of multiplication as 'factors' and the particular number named as the 'product' of the factors. We say, for example:

'Two times three equals six'."

"I notice two patterns that will help me. It seems that order in considering the factors is not important."

"Right. This is called the 'Commutative Law of Multiplication' and we might describe the pattern as:

$$x \cdot y = y \cdot x \quad \text{or} \quad xy = yx$$

What other patterns do you notice?"

"If one of the factors is 1, then the product and the other factor are the same number. I suppose you might point out the pattern as:

$$x \cdot 1 = 1 \cdot x = x$$

"

"Right. Mathematicians say that 1 is the Identity Element for Multiplication."

Consider the following master plan that will help X-Nighyun see the really big though simple patterns of arithmetic.

Numbers Have Many Names

Addition

You can add any two numbers a and b and there is only one correct answer.

(1) $a + b = x$

For all numbers a and b

(3) $a + b = b + a$

For all numbers a, b and c

(5) $a + (b + c) = (a + b) + c$

Multiplication

You can multiply any two numbers a and b and there is only one correct answer.

(2) $a \cdot b = x$

For all numbers a and b

(4) $a \cdot b = b \cdot a$

For all numbers a, b and c

(6) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

For all numbers a, b and c

(7) $a \cdot (b + c) = a \cdot b + a \cdot c$

For all numbers a and b, there is a number x such that

(8) $a - b = x$

There is a number 0 such that for every a,

(10) $a + 0 = a$

For all numbers a and b, there is a number x such that

(9) $a \div b = x$ (If $b \neq 0$)

There is a number 1 such that for every a

(11) $a \cdot 1 = a$

The product of two numbers is 0 if and only if at least one of the factors is 0.

(12) If $a \cdot b = 0$, then $a = 0$ or $b = 0$

Patterns (1), (2), (3), (4), (5), (10) and (11) have already been considered.

As soon as some of the addition and multiplication facts are understood, we can explain what we mean by "Numbers Have Many Names."

We gave X-Nighyun nine combinations each for addition and multiplication. Already he can see that some numbers obviously have several names.

$2 = 1 \times 2 = 2 \times 1 = 1 + 1$

$3 = 1 \times 3 = 3 \times 1 = 1 + 2 = 2 + 1$

$4 = 1 + 3 = 3 + 1 = 2 + 2 = 2 \times 2$

$6 = 3 + 3 = 2 \times 3 = 3 \times 2$

We have given X-Nighyun some facts of multiplication, but there has been no discussion about how he can learn other facts by himself. He doesn't know yet what we mean by "multiplication."

There are several ways we could explain this. However, suppose we get him to understand what we mean when we arrange things in "rows and columns."

"Have you got a lot of things with you up there in your space ship?"

"Sure: lots of sandwiches and books and dials and . . ."

"Well, arrange four books in a row. Now make two more rows — three in all. How many books?"

"Twelve."

"Right. So we say that three times four equals twelve. If we think of 3 and 4 and the idea of multiplication, we say that the product of 3 and 4 is 12."

"That's easy. So 3 times 5 must be . . . 15. What do I do when I run out of books?"

"You'll have to work that problem out for yourself."

"But how about two times three, times five?"

"Arrange the books in two rows and three columns. Now call that a layer. Put 4 more layers on top — 5 layers in all. How many books?"

"Thirty!"

"Now find out for yourself whether the Associative Law applies to multiplication. Is it true that

$$2 \times (3 \times 5) = (2 \times 3) \times 5 \ ?$$"

"That's true! So the Commutative and Associative Laws hold for multiplication. Together they tell me that I can deal with factors in any order."

"You are right again."

"Is there any connection between addition and multiplication?"

"Yes, a very important rule called the Distributive Law is suggested by the following:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

I wish I could draw you some pictures to illustrate the idea, but you'll have to try some examples to see that it is a sensible rule."

$2\,(3+4) = 2 \cdot 7 = 14$

$2\,(3+4) = 2 \cdot 3 + 2 \cdot 4 = __ + __ = __$

$7\,(8+2) = 7 \cdot 10 = __$

$7\,(8+2) = 7 \cdot 8 + __ = __ + __ = __$

"We use this Distributive Law to simplify multiplication. We don't remember the product of 7 and 86. We change 86 to one of its other names — 80 + 6 — and then we say

$7(80+6) = 7 \cdot 80 + 7 \cdot 6 = __ + __ = __$"

It will also be necessary to explain that we can find a missing addend when one addend and the sum are given. If a is any sum and b is a given addend, we can find the other addend x.

$$a - b = x$$

Also, if a is any given product and b is a known factor, we can find the missing factor x.

$$a \div b = x$$

We will explain that finding a missing addend is called "subtraction" and that finding a missing factor is called "division."

Finally, we will explain that if the product of a pair of factors is zero, then one or both of the factors must be zero.

If $a \cdot b = 0$, then $a = 0$ or $b = 0$

X-Nighyun Raises Two Questions

"You hurried over a couple of questions. I understand that $a - b = x$ if a is a number larger than the number for b. You call that subtraction — looking for the missing addend. But suppose the sum is smaller than the given addend — can you still subtract? I seem to run out of counting numbers."

X-Nighyun gave this set of examples:

$2 - 0 = \underline{2}$	$5 - 4 = \underline{1}$
$2 - 1 = \underline{1}$	$5 - 5 = \underline{0}$
$2 - 2 = \underline{0}$	$5 - 6 = \underline{\quad}$?
$2 - 3 = \underline{\quad}$?	$5 - 7 = \underline{\quad}$?

"I forgot to explain that when this problem presents itself, we must invent some new numbers. We think of 'above and below zero' or 'to the right and left of zero' as on a numberline or 'forward and backward from a starting point.'

"The numbers that seem to behave like counting numbers are called 'positive numbers' and those in the opposite direction from zero are called 'negative numbers.' We say that each positive number has an 'opposite' — a negative number. The opposite of positive six — the opposite of $^{+}6$ is $^{-}6$ and we say that

$^{+}a + {}^{-}a = 0$

and in your examples, we say that

$^{+}2 - {}^{+}3 = {}^{-}1$, $^{+}5 - {}^{+}6 = {}^{-}1$, $^{+}5 - {}^{+}7 = \underline{\quad}$

"With this start you ought to be able to extend each of the patterns in your two sets of examples in either direction. It may sound peculiar, but we say that since zero is not in any sense on either side of zero, we don't assign the name positive or negative to this number. It's simply 'zero'."

"What you say makes some sense, and I will experiment with these new numbers and let you know my problems. But what about what you call division? It's 'looking for the missing factor' but is it always possible? What about

$2 \div 3 = \underline{\quad}$ and $7 \div 2 = \underline{\quad}$?

Your new positive and negative numbers don't help me any there."

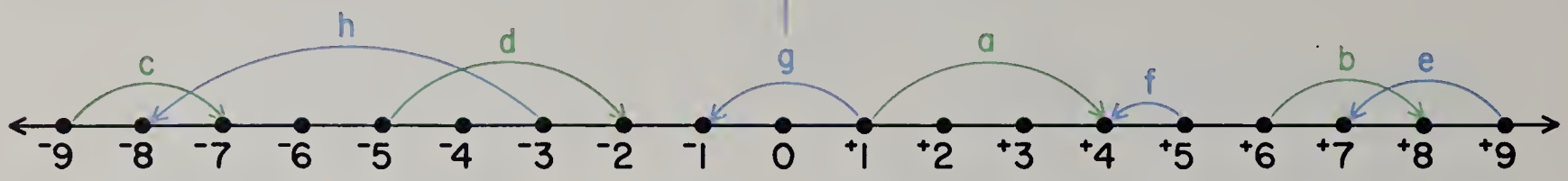
"That's right. So we have to invent more new numbers. If you had 2 sandwiches to divide among 3 Martians, how much would each get? We invent $\frac{2}{3}$ as the answer to that question."

"I suppose you are going to tell me that $7 \div 2 = \frac{7}{2}$ . . . seven divided by two equals seven divided by two!"

"Almost as simple as that. If 2 Martians share 7 sandwiches, then each would get 7 halves or 3 whole sandwiches and one-half of another. We call that 3 and $\frac{1}{2}$."

$a - b = x$. . . Subtraction is always possible! . . . (The "principle of closure" for subtraction)

As X-Nighyun experiments with some of the new numbers — positive and negative numbers and zero — we ought to be prepared to answer the questions that will arise. We might explain to him what we mean by a number line.



We can simplify our language a little by referring to $^{+}2$ as "two" and $^{-}2$ as "the opposite of two." Of course, if you prefer, you can speak of "positive two" and of "negative two."

These new numbers behave on a number line much as the counting numbers do. We begin by thinking of addition as a jump to the right (see green arrows).

a. $^{+}1 + {}^{+}3 = {}^{+}4$ c. $^{-}9 + {}^{+}2 = {}^{-}7$

b. $^{+}6 + {}^{+}2 =$ ___ d. $^{-}5 + {}^{+}3 =$ ___

We can also begin thinking of subtraction as a jump in the opposite direction . . . if the known addend is not the opposite of a number.

e. $^{+}9 - {}^{+}2 =$ ___ g. $^{+}1 - {}^{+}2 =$ ___

f. $^{+}5 - {}^{+}1 =$ ___ h. $^{-}3 - {}^{+}5 =$ ___

other examples:

i. $^{+}2 + {}^{+}2 =$ ___ j. $^{+}2 - {}^{+}2 =$ ___

k. $^{-}4 + {}^{+}3 =$ ___ l. $^{-}4 - {}^{+}3 =$ ___

m. $^{+}1 + {}^{+}8 =$ ___ n. $^{+}1 - {}^{+}8 =$ ___

Follow the Patterns!

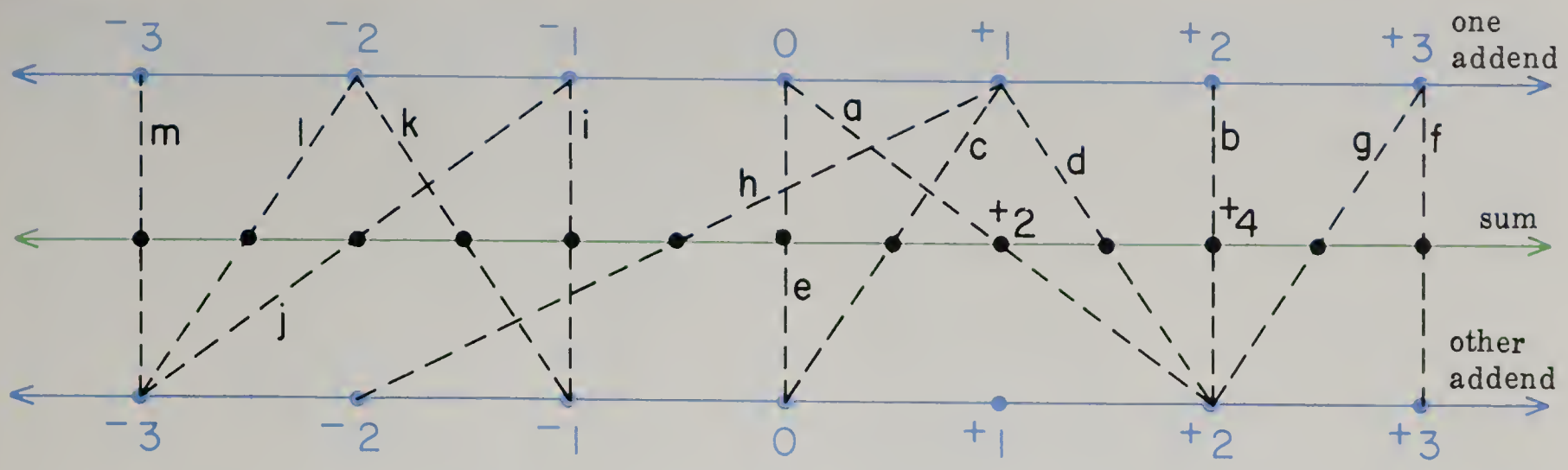
We know enough to get some patterns started. Where do the patterns lead?

I	II	III	IV
$^{+}3 + {}^{+}2 =$ ___	$^{+}3 - {}^{+}2 =$ ___	$^{-}3 + {}^{+}2 =$ ___	$^{-}3 - {}^{+}2 =$ ___
$^{+}3 + {}^{+}1 =$ ___	$^{+}3 - {}^{+}1 =$ ___	$^{-}3 + {}^{+}1 =$ ___	$^{-}3 - {}^{+}1 =$ ___
$^{+}3 + 0 =$ ___	$^{+}3 - 0 =$ ___	$^{-}3 + 0 =$ ___	$^{-}3 - 0 =$ ___
$^{+}3 + {}^{-}1 =$ ___	$^{+}3 - {}^{-}1 =$ ___	$^{-}3 + {}^{-}1 =$ ___	$^{-}3 - {}^{-}1 =$ ___
$^{+}3 + {}^{-}2 =$ ___	$^{+}3 - {}^{-}2 =$ ___	$^{-}3 + {}^{-}2 =$ ___	$^{-}3 - {}^{-}2 =$ ___
$^{+}3 + {}^{-}3 =$ ___	$^{+}3 - {}^{-}3 =$ ___	$^{-}3 + {}^{-}3 =$ ___	$^{-}3 - {}^{-}3 =$ ___

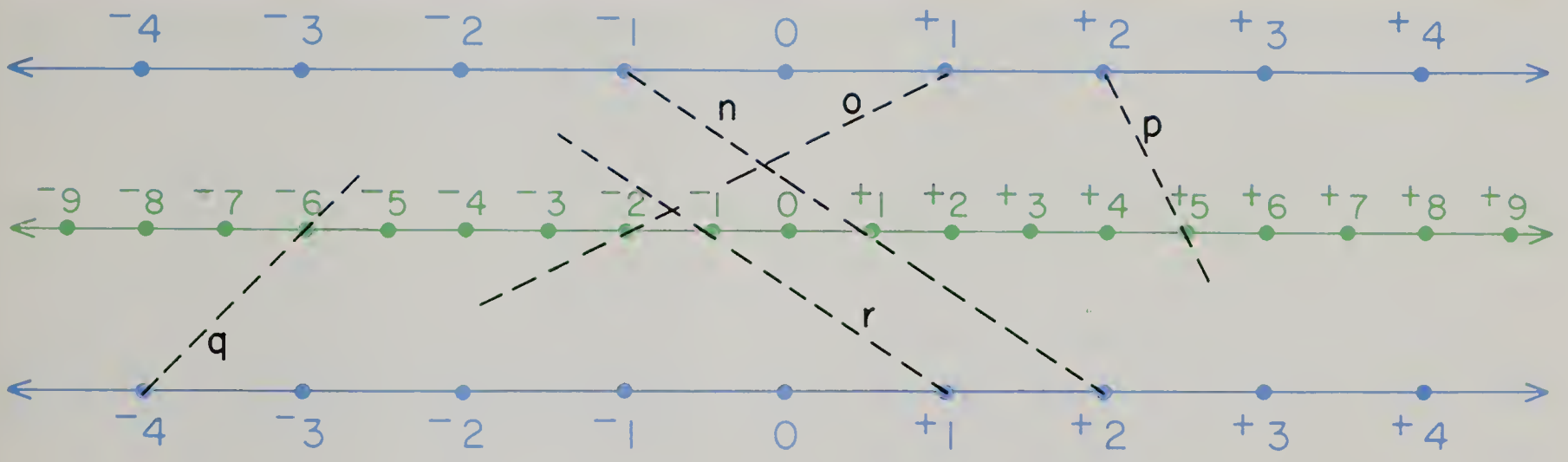
These results will help on the triple number line on the next page.

Stretch your mental muscles.

x	2	4	8	$\frac{1}{2}$	.5		$\frac{1}{3}$					7	
y	5	7		$\frac{1}{2}$			$\frac{1}{4}$				1		
$x + y$	7		11			10		16	$\frac{1}{2}$				$1\frac{1}{2}$
$x \cdot y$ or xy	10				3.5	21		63	$\frac{1}{16}$	49	1	0	$\frac{9}{16}$



lines	a	b	c	d	e	f	g	h	i	j	k	l	m
one addend	0	+2	+1	+1									
other addend	+2	+2	0										
sum	+2	+4											



Draw lines (and label them) to find the missing addend or sum.

lines	n	o	p	q	r	s	t	u	v	w	x	y	z
one addend	−1	+1	+2			−4	−4		−4	+1			+1
sum	+1	−2	+5	−6	−1	−8	−7	−6	−5	−1	0	+1	+4
other addend	+2			−4	+1			−2			−1	0	

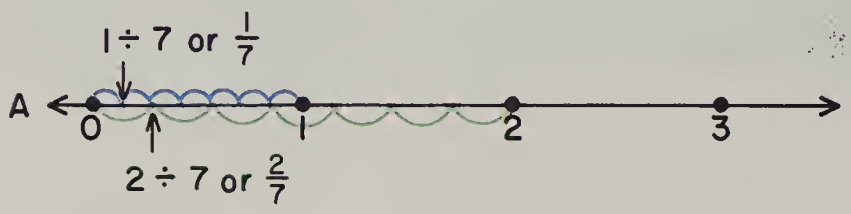
Find the results in the examples below.
Use a straight-edge, but do not draw lines.

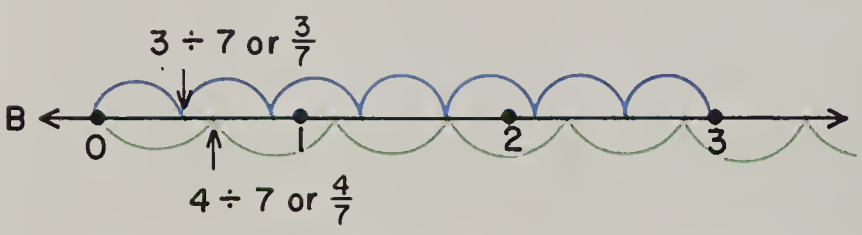
examples	A	B	C	D	E	F	G	H	I	J	K	L	M
sum	−1	−1	−1	−1	−1	0	0	0	+1	+1	+1	+1	+1
one addend	−3	−2	−1	0	+1	+2	0	−2	−1	0	+1	+2	+3
other addend	+2												

X-Nighyun knows about "rows and columns" and can find that $3 \times 7 = 21$ because he can arrange sandwiches so there are 3 rows and 7 columns . . . and count to 21. And, so, he knows that $21 \div 3 =$ ___ asks for a missing factor . . . 7 in this case.

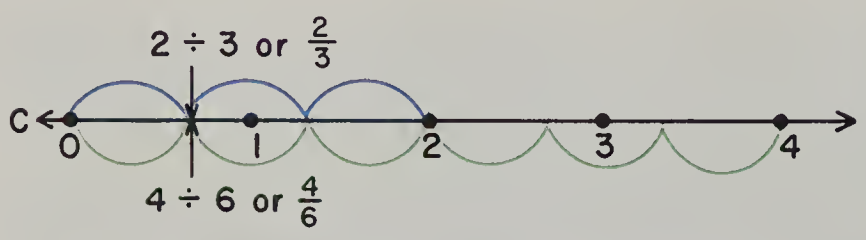
But $7 \div 2 =$ ___ still bothers him because he doesn't know any answer to $2 \times$ ___ $= 7$.

Suppose we explain that we invent numbers to label points between those on the number line labeled with the numbers of counting.



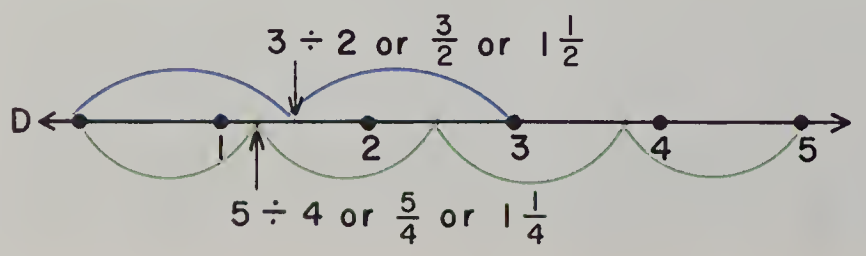


Divide the distance from 1 on the number line in 7 equal parts. This suggests the idea of $1 \div 7$ or $\frac{1}{7}$. Divide the distance from 2 in 7 parts. One of these suggests the idea of 2 divided by 7 or $\frac{2}{7}$ . . . etc.



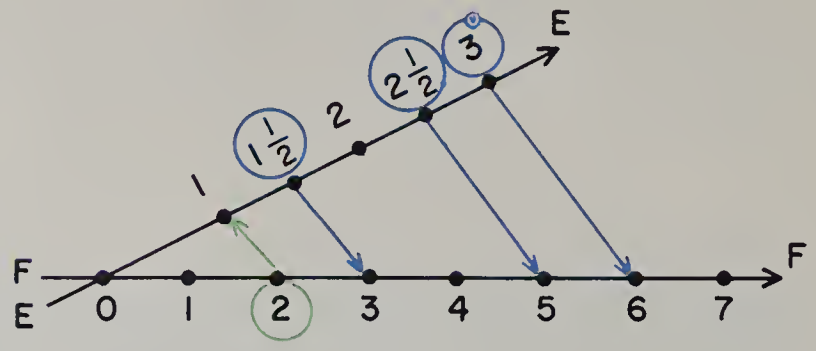
By this procedure, we sometimes find that points have more than one label; $2 \div 3$ and $4 \div 6$ name the same point.

Similarly, we can name points that are not between 0 and 1.



Suppose we tell X-Nighyun about an interesting machine he can use to see multiplication from a new viewpoint.

Draw two number lines that cross — and with "0" labeling the point of intersection.



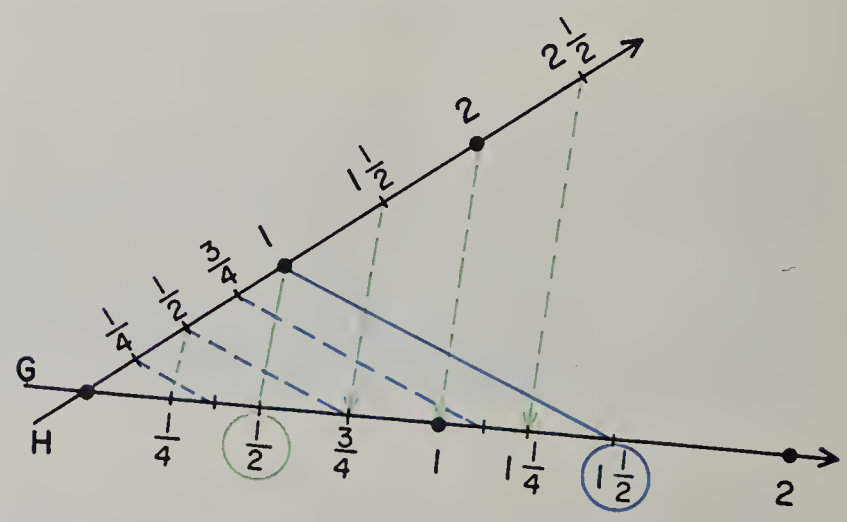
If you want to multiply by 2, for example, draw a line connecting the point labeled 2 on one number line (F) to the point labeled 1 on the other number line (E).

Next, draw a line from the point on line E labeled by any other factor — draw it parallel to the first line.

$2 \cdot 1\frac{1}{2} =$ ____ $\quad 2 \cdot 2\frac{1}{2} =$ ____

$2 \cdot 3 =$ ____ $\quad 2 \cdot 2 =$ ____

Here is another example. Notice that the angle between the lines has been changed and so has the scale on each line.



Suppose you want to multiply by $\frac{1}{2}$. Connect the point on G labeled $\frac{1}{2}$ to the point labeled 1 on H (solid green line). And, if you want to multiply, in other examples by $1\frac{1}{2}$, connect the point on G so labeled to the point labeled 1 on number line H (solid blue lines).

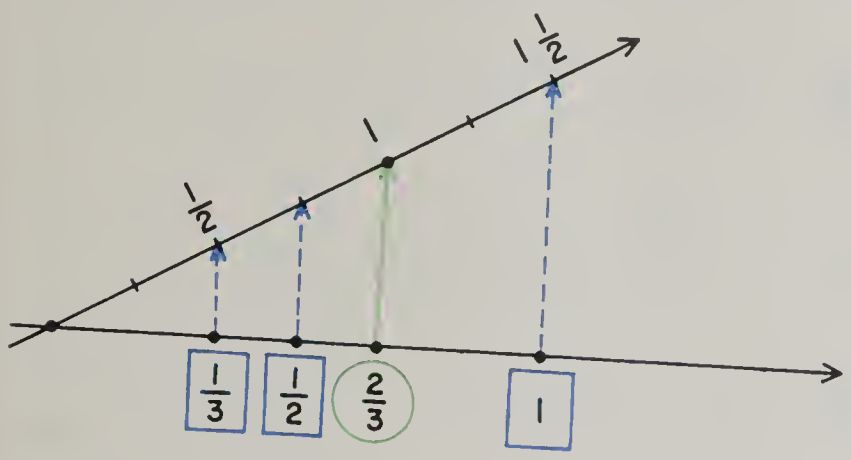
If $\frac{1}{2}$ is one factor, locate the point on H labeled with the same number as the other factor. Draw a line parallel to the solid green line from that point to line G. It will indicate the product.

If $1\frac{1}{2}$ is one factor, follow the same procedure as indicated by the blue lines.

$\frac{1}{2} \cdot 1\frac{1}{2} =$ $\quad\quad$ $\frac{1}{2} \cdot 2 =$ $\quad\quad$ $\frac{1}{2} \cdot 2\frac{1}{2} =$

$1\frac{1}{2} \cdot \frac{3}{4} =$ $\quad\quad$ $1\frac{1}{2} \cdot \frac{1}{2} =$ $\quad\quad$ $1\frac{1}{2} \cdot \frac{1}{4} =$

Now, we can use this sketch to help find a "missing factor" in division.



Connect the point labeled with the same number as the known factor (green circle) to the point labeled 1 on the other line. With a parallel line from the point labeled as the product (blue rectangle), find a point on the other line. The point located will be labeled as the other factor.

$\boxed{\frac{1}{2}} \div \left(\frac{2}{3}\right) =$ $\quad\quad$ $\boxed{\frac{1}{3}} \div \left(\frac{2}{3}\right) =$

$\boxed{1} \div \left(\frac{2}{3}\right) =$ $\quad\quad$ $\boxed{\frac{2}{3}} \div \left(\frac{2}{3}\right) =$

About this time, X-Nighyun might ask a very important question.

"When on earth would you ever use a number in that way — dividing by a fraction?"

"Often — and usually in measurement. Here are examples:"

a. How many $\frac{1}{2}$ gallons are there in $7\frac{1}{2}$ gallons? Ans.: _____

b. Ruth had $4\frac{1}{2}$ feet of ribbon and Mary had 2 yards of ribbon. Each cut her ribbon into pieces $\frac{3}{4}$ of a foot long. Ruth had ____ pieces and Mary had _____ pieces.

c. How many $\frac{1}{3}$ hour periods are there between 8:20 A.M. and 12:40 P.M.?

____

d. How many 45 minute periods in $4\frac{1}{2}$ hours? _____

e. Mr. Brown paid each helper $3\frac{1}{2}$ dollars. In all, he paid out $24\frac{1}{2}$ dollars. How many helpers were there? _____

f. After a space ship has travelled 100,000 miles of its 250,000 mile trip to the moon, has it gone more or less than $\frac{3}{8}$ of the way? _____

g. How many $2\frac{1}{4}$ foot lengths of pipe can be cut from a 27 foot length? ____

h. How much cloth can be purchased with 75¢ if the cloth costs 50¢ per yard?

_____

i. How many tiles $\frac{3}{4}$ of a foot by $\frac{3}{4}$ of a foot are required to cover a square yard?

_____

$a + b$					24	50		1.9		1			
$a - b$			7					.5	$\frac{1}{8}$	$\frac{2}{5}$	38	5.4	$\frac{1}{6}$
a	4	25	7	8		30	$\frac{1}{2}$						
b	1	16			9		$\frac{1}{4}$						
$2(a+b)$				16					$3\frac{3}{4}$		200	13.2	$1\frac{2}{3}$

"Tell me more about multiplying these numbers you call fractions."

"Well, it helps a little to get the idea if you use 'of' and 'times' as meaning almost the same thing.

"A half of a half-dollar is a quarter of a dollar; one-third of a half a pie is one-sixth of a pie; three-fifths of a half-mile is three-tenths of a mile; three-fourths of a half-pie is three-eighths of a pie. We can express these ideas in the form of multiplication.

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\qquad$ $\frac{1}{3} \cdot \frac{1}{2} =$

$\frac{3}{5} \cdot \frac{1}{2} =$ $\qquad$ $\frac{3}{4} \cdot \frac{1}{2} =$

"The pattern this suggests is:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \text{ or } \frac{ac}{bd}$$

"If this is true, then the following must be true:

$\frac{3}{1} \cdot \frac{4}{1} = \frac{12}{1}$ $\qquad$ $\frac{6}{1} \cdot \frac{1}{2} = \frac{6}{2} = 3$

$\frac{4}{1} \cdot \frac{1}{4} = — =$ $\qquad$ $\frac{1}{2} \cdot \frac{10}{1} = — =$

$\frac{2}{1} \cdot \frac{3}{2} = — =$ $\qquad$ $\frac{1}{2} \cdot \frac{2}{3} = — =$

$\frac{5}{2} \cdot \frac{2}{1} = — =$ $\qquad$ $\frac{1}{4} \cdot \frac{1}{4} = —$

$\frac{1}{3} \cdot \frac{2}{3} = —$ $\qquad$ $\frac{1}{4} \cdot \frac{3}{4} = —$

"So, if we follow the rule above, we keep producing results that seem to make a lot of sense.

"Here are some other rules we will use as we move ahead.

$$\frac{ak}{bk} = \frac{a \cdot k}{b \cdot k} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b}$$

"The idea here is that every fraction can be written in the form of $\frac{ak}{bk}$ where k is the largest common factor; and that $\frac{ak}{bk}$ is $\frac{a}{b}$ . . . with a and b having no common factor. Here are examples:

$\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3} \cdot \frac{3}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3}$

$\frac{15}{35} = \frac{3 \cdot 5}{\cdot 5} = — \cdot \frac{5}{5} = — \cdot 1 = —$

$\frac{24}{60} = \frac{\cdot 12}{\cdot} = — \cdot — = — \cdot 1 = —$

$\frac{21}{49} = \frac{\cdot}{\cdot} = — \cdot — = — \cdot 1 = —$

"Some people refer to this idea as 'reducing a fraction to its lowest terms.'

"Further, there is a useful pattern we describe as:

$$\frac{a}{b} \cdot x = 1$$

"This means that no matter what number (including any fraction) is taken as one factor, a second factor can be chosen so that their product is 1."

$\frac{2}{1} \cdot \frac{1}{2} = \frac{2}{2} = 1$ $\qquad$ $\frac{1}{3} \cdot — = \frac{3}{3} = 1$

$\frac{3}{4} \cdot — = — = 1$ $\qquad$ $\frac{2}{9} \cdot — = — = 1$

"What's so important about that? It doesn't seem either important or interesting."

"I'll show you.

"Think of any division fact such as $3 \div \frac{1}{2} = 6$. Now think of any number and multiply both the product and either factor by that number. Try 4, 8 and 20."

$3 \cdot 4 \div \frac{1}{2} \cdot 4 = 6$ or $3 \cdot 4 \div \frac{1}{2} = 6 \cdot 4$

$12 \div 2 = 6$ or $12 \div \frac{1}{2} = ___$

$3 \cdot 8 \div \frac{1}{2} \cdot ___ = 6$ or $3 \cdot 8 \div \frac{1}{2} = 6 \cdot 8$

$___ \div ___ = 6$ or $___ \div \frac{1}{2} = ___$

$3 \cdot 20 \div \frac{1}{2} \cdot 20 = 6$ or $3 \cdot 20 \div \frac{1}{2} = 6 \cdot 20$

$___ \div ___ = ___$ or $___ \div __ = ___$

"Now I don't consider that is a new idea. I can put the pattern this way. You start with a fact in the form of

$$a \div b = m$$

Then you tell me that I can multiply the product and a factor by any number x. Or

$$a \cdot x \div b \cdot x = m$$

Can't the whole idea be written as

If $\frac{a}{b} = m$ then $\frac{a \cdot x}{b \cdot x} = m$?"

"That is right. But let's use the pattern to see what sense we can make out of

$$\frac{7}{8} \div \frac{2}{3} = ?$$

"Will you agree with me that

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot x \div \frac{2}{3} \cdot x \text{ ?}"$$

"Of course. That's just what we've been talking about."

"All right. I'm going to select $\frac{3}{2}$ as x and I will write

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} \div \frac{2}{3} \cdot \frac{3}{2}"$$

"Fair enough. But, why?"

"Because $\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1$, and I can now write

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{2}{3} \div 1 = \frac{7}{8} \cdot \frac{2}{3}$$

because division of any number by 1 leaves the number unchanged. Now, I use the pattern

$$\frac{a}{b} \cdot \frac{x}{y} = \frac{ax}{by}$$

and I write

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \cdot \frac{3}{2} = \frac{21}{16} = 1\frac{5}{16}"$$

"Great Martian winter! That was a neat trick! . . . and it sure makes sense. Let me try that argument on some examples.

$$\frac{6}{1} \div \frac{1}{2} = \frac{6}{1} \cdot \frac{2}{1} \div \frac{1}{2} \cdot \frac{2}{1} = \frac{6}{1} \cdot \frac{2}{1} \div 1 = \frac{6}{1} \cdot \frac{2}{1} = 12$$

"Now I'm going to skip some steps.

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \cdot \frac{4}{1} = \frac{4}{2} = 2$$

$$\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \cdot _ = _ =$$

$$\frac{4}{2} \div \frac{2}{3} = _ \cdot _ = _ =$$

$$\frac{2}{3} \div \frac{7}{8} = _ \cdot _ = _$$

"Now I realize that is both important and interesting."

"You sure have my head swimming. Please go back and outline your over-all plan to me again. Now that I see where you are going, I will be able to understand more."

This time, you decide, we will use other letters than a, b, and c to point out the patterns so X-Nighyun won't think that the selection of letters has anything to do with the patterns.

Numbers Have Many Names!

Addition

(1) $r + s = t$ Principle of Closure . . . addition is always possible.

(3) $r + s = s + r$ Addition is Commutative.

(5) $r + (s + t) = (r + s + t)$ Addition is Associative.

Multiplication

(2) $rs = t$ Principle of Closure . . . multiplication is always possible.

(4) $rs = sr$ Multiplication is Commutative.

(6) $r\,(s\,t) = (rs)\,t$ Multiplication is Associative.

(7) $r\,(s + t) = rs + rt$ Multiplication is Distributive over addition.

(8) $r + 0 = r$ Zero is the Identity Element for addition.

(10) $r - s = t$ Subtraction is always possible.

(9) $r \cdot 1 = r$ One is the Identity Element for multiplication.

(11) $r \div s = t$ Division is always possible if $s \neq 0$.

(12) If $rs = 0$, then $r = 0$ or $s = 0$

X-Nighyun had some questions.

"I notice that all your patterns are concerned either with addition or with multiplication — except one — that multiplication is Distributive over addition. Is that right?"

"Exactly. We use that idea so we do not need to remember the facts of multiplication beyond 10×10. If we need to know 9×18, we use another name for 18 — we call it $10 + 8$, and then use the Distributive idea:

$9(10 + 8) = 9 \cdot 10 + 9 \cdot 8 = 90 + 72 = 162$"

"Another question I need help with is concerned with rules (1), (2), (10) and (11). If I summarize them properly, they say that it is always possible to add, subtract, multiply and divide."

"That is true only with the system of numbers that includes the idea of fractions and the idea of negative and positive numbers or numbers and their 'opposites'."

"It would help me if you took these rules one at a time and explained to me a few examples that I will give."

$x + y = z$ Addition is always possible.

Examples given by X-Nighyun:

$28 + 35 =$	$3.5 + .72 =$	$\frac{2}{3} + \frac{7}{8} =$
$^{-}3 + ^{-}4 =$	$^{-}3 + ^{+}4 =$	$^{+}3 + ^{-}4 =$

"Since numbers have many names, we can write 28 + 35 as follows:

$20 + 8 + 30 + 5$

"We select the terms 8 and 30 and then apply the Commutative principle:

$20 + (8 + 30) + 5 = 20 + (30 + 8) + 5$

"Next, we can apply the Associative principle to the first three terms:

$20 + (30 + 8) + 5 = (20 + 30) + 8 + 5$

"Then we find other names for 20 + 30 and for 8 + 5 and write:

$(20 + 30) + 8 + 5 = 50 + 13$

"We find another name for 13 and use the Associative rule again:

$50 + (10 + 3) = (50 + 10) + 3 = 60 + 3 = 63$

"Of course, we shorten this long business and think of what's happening. We look at it this way:"

$$\begin{array}{r} 28 \\ 35 \\ \hline 63 \end{array} \rightarrow \begin{array}{r} 20 + 8 \\ 30 + 5 \\ \hline 50 + 13 \end{array} \rightarrow \begin{array}{r} 50 + 0 \\ 10 + 3 \\ \hline 60 + 3 = 63 \end{array}$$

Try some examples of addition using the form suggested here.

"The next two examples are essentially the same if you write them both as addition of common fractions.

$\frac{35}{10} + \frac{72}{100} =$ and $\frac{2}{3} + \frac{7}{8} =$

"We want common denominators so we multiply certain names for one to accomplish our purpose."

$\frac{35}{10} \cdot \frac{10}{10} + \frac{72}{100} =$

$\frac{350}{100} + \frac{72}{100} = \frac{422}{100}$

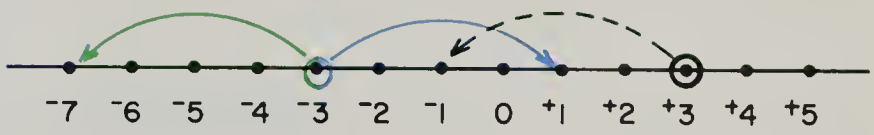
$= 4.22$

$\frac{2}{3} \cdot \frac{8}{8} + \frac{7}{8} \cdot \frac{3}{3} =$

$\frac{16}{24} + \frac{21}{24} = \frac{37}{24}$

$= 1\frac{13}{24}$

"The next three examples are related. Think of a number line. In each example, start at the point labeled as the first addend. Then take a jump as indicated by the other addend — but the direction is indicated by the sign; positive to the right, negative in the opposite direction."



$^{-}3 + ^{-}4 =$ ____	$^{-}3 + ^{+}4 =$ ____
$^{+}3 + ^{-}4 =$ ____	$^{+}3 + ^{+}4 =$ ____
$^{-}4 + ^{+}3 =$ ____	$^{+}4 + ^{+}3 =$ ____
$^{-}2 + ^{-}1 =$ ____	$^{-}2 + ^{+}1 =$ ____
$^{-}5 + ^{+}5 =$ ____	$^{+}3 + ^{-}3 =$ ____

m	48	$\frac{1}{2}$	2			.2		3		3		$\frac{1}{6}$	
n	4	$\frac{1}{2}$	$\frac{1}{3}$	2	$\frac{3}{4}$		6		.1		17		$\frac{2}{3}$
$m \div n$				6	4	.5				$\frac{1}{2}$			$1\frac{5}{16}$
$2mn$								30	.2		4,046	$\frac{1}{9}$	
$2n + m$	56						29						

$p - q = r$ Subtraction is always possible.

Examples submitted by X-Nighyun:

$53 - 17 =$ ____ $\quad 4 - 7 =$ ____

$^{+}9 - {}^{-}7 =$ ____ $\quad {}^{-}3 - {}^{+}8 =$ ____

"Here, we will take a detour. Are you convinced that addition is always possible after we invent positive and negative numbers?"

"I think I am ready to believe that. Of course, I want to try many more examples."

"Fine. Then I am going to show you how you can write your examples in the form of addition. That's all I need to do, because you are convinced that addition is always possible."

"If you can, that would be another surprise for me. I would have to agree that you had proved your argument."

"Good. First, when the sum and one addend is given, it will be in one of the following four forms:

$^{+}a - {}^{+}b =$ ____ $\quad {}^{+}a - {}^{-}b =$ ____

$^{-}a - {}^{+}b =$ ____ $\quad {}^{-}a - {}^{-}b =$ ____

"Remember that in place of the letters in these pattern indicators, I can put any numbers I like, including fractions; so all your examples are illustrations of one of these forms. I am now going to try to convince you that

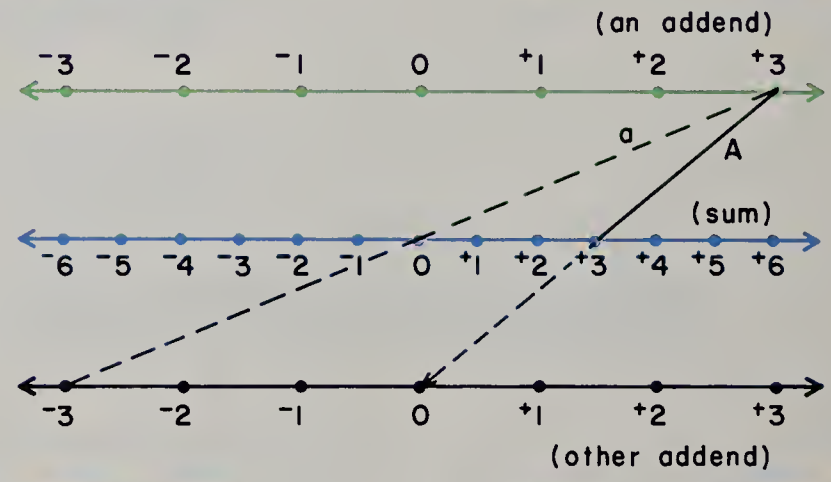
$$\begin{aligned} {}^{+}a - {}^{+}b &= {}^{+}a + {}^{-}b \\ {}^{+}a - {}^{-}b &= {}^{+}a + {}^{+}b \\ {}^{-}a - {}^{+}b &= {}^{-}a + {}^{-}b \\ {}^{-}a - {}^{-}b &= {}^{-}a + {}^{+}b \end{aligned}$$

"And you will notice that this is a shorthand way of saying that expressions calling for subtraction can be rewritten in the form of addition."

"But, is it true that $^{+}a + {}^{-}b$ names the same number as $^{+}a - {}^{+}b$?"

"Complete the patterns that are evident in the following. Use the triple number line to help reach reasonable conclusions."

A	$^{+}3 - {}^{+}3 =$ ___		a	$^{+}3 + {}^{-}3 =$ ___	
B	$^{+}3 - {}^{+}2 =$ ___		b	$^{+}3 + {}^{-}2 =$ ___	
C	$^{+}3 - {}^{+}1 =$ ___		c	$^{+}3 + {}^{-}1 =$ ___	
D	$^{+}3 - 0 =$ ___		d	$^{+}3 + 0 =$ ___	
E	$^{+}3 - {}^{-}1 =$ ___		e	$^{+}3 + {}^{+}1 =$ ___	
F	$^{+}3 - {}^{-}2 =$ ___		f	$^{+}3 + {}^{+}2 =$ ___	
G	$^{+}3 - {}^{-}3 =$ ___		g	$^{+}3 + {}^{+}3 =$ ___	



H	$^{-}3 - {}^{+}3 =$ ___		h	$^{-}3 + {}^{-}3 =$ ___	
I	$^{-}3 - {}^{+}2 =$ ___		i	$^{-}3 + {}^{-}2 =$ ___	
J	$^{-}3 - {}^{+}1 =$ ___		j	$^{-}3 + {}^{-}1 =$ ___	
K	$^{-}3 - 0 =$ ___		k	$^{-}3 + 0 =$ ___	
L	$^{-}3 - {}^{-}1 =$ ___		l	$^{-}3 + {}^{+}1 =$ ___	
M	$^{-}3 - {}^{-}2 =$ ___		m	$^{-}3 + {}^{+}2 =$ ___	
N	$^{-}3 - {}^{-}3 =$ ___		n	$^{-}3 + {}^{+}3 =$ ___	

"Now, notice each pair of examples labeled with the same letter — one capital and one small.

(A,a) $^{+}3 - {}^{+}3 = 0$ and $^{+}3 + {}^{-}3 = 0$

"Examine each pair. In each case, two different expressions name the same number. So, we can say:"

(A,a) $^{+}3 - {}^{+}3 = {}^{+}3 + {}^{-}3$

(G,g) $^{+}3 - {}^{-}3 =$ ___ + ___

(H,h) $^{-}3 - {}^{+}3 =$ ___ + ___

(N,n) $^{-}3 - {}^{-}3 =$ ___ + ___

"I am impressed. This certainly suggests that you can change any expression involving subtraction to an equivalent expression involving addition."

"Of course, this is far from a proof, but it is a beginning.

"To help further, let's think of $^{+}3$ as earning \$3 and $^{-}3$ as incurring a debt of \$3. Also, think of addition as 'receiving' and subtraction as 'getting rid of' or 'spending.' With this interpretation, we have the following.

"Also, we will interpret a positive answer as 'to the good' and a negative answer as 'worse off.' "

$^{+}3 - {}^{+}3$...You earn \$3 and you get rid of \$3. You are \$0 to the good.

$^{+}3 - {}^{-}3$...You earn \$3 and you get rid of a \$3 debt. You are \$6 to the good.

$^{-}3 - {}^{+}3$...You incur a \$6 debt and get rid of \$3. You are worse off by \$6.

$^{-}3 - {}^{-}3$...You incur a debt of \$3 and get rid of a \$3 debt. You are \$0 to the good.

With a similar interpretation, write about the situation suggested by the following headlines after they are completed:

$^{+}4 + {}^{+}5 =$

$^{+}4 - {}^{+}7 =$

$^{-}8 + {}^{-}10 =$

$^{-}15 + {}^{+}21 =$

$^{-}18 - {}^{-}8 =$

$^{-}1 - {}^{-}1 =$

$^{+}\frac{1}{2} + {}^{-}\frac{1}{2} =$

$^{-}\frac{3}{4} + {}^{-}\frac{1}{2} =$

$n \cdot c = h$ Multiplication is always possible.

X-Nighyun had only a limited question about this matter.

"You know that positive numbers behave very much like the counting numbers, so you would expect that

$$^{+}2 \cdot {}^{+}5 = {}^{+}10$$

"Suppose that the thermometer fell 5° and we suggest this idea by $^{-}5$. Again, suppose that during each of two hours, the thermometer fell 5°. We might express this idea as

$$2 \cdot {}^{-}5 = {}^{-}10$$

"Or, it fell 10°. Since 2 and $^{+}2$ behave very much alike, we say that

$$^{+}2 \cdot {}^{-}5 = {}^{-}10$$

"If the thermometer fell 2° during each of 5 hours, we might similarly suggest this idea as

$$^{-}2 \cdot 5 = {}^{-}10 \quad \text{or} \quad 5 \cdot {}^{-}2 = {}^{-}10$$

"We would want this to be true so the Commutative Law would hold. This would lead us to say:

$$^{-}2 \cdot 5 = 5 \cdot {}^{-}2$$

"The hardest problem is to decide what we might sensibly mean by

$$^{-}2 \cdot {}^{-}5 = ?$$

"Consider two groups of examples. We have already agreed on the meaning of most of the expressions. But how can we fill in the blanks in the blue-tinted areas?"

$^{+}2 \cdot {}^{+}3 =$ ___	$^{-}2 \cdot {}^{+}3 =$ ___
$^{+}2 \cdot {}^{+}2 =$ ___	$^{-}2 \cdot {}^{+}2 =$ ___
$^{+}2 \cdot {}^{+}1 =$ ___	$^{-}2 \cdot {}^{+}1 =$ ___
$^{+}2 \cdot 0 =$ ___	$^{-}2 \cdot 0 =$ ___
$^{+}2 \cdot {}^{-}1 =$ ___	$^{-}2 \cdot {}^{-}1 =$ ___
$^{+}2 \cdot {}^{-}2 =$ ___	$^{-}2 \cdot {}^{-}2 =$ ___
$^{+}2 \cdot {}^{-}3 =$ ___	$^{-}2 \cdot {}^{-}3 =$ ___

"I can see that anyone would hope to be able to say that

$$^{-}2 \cdot {}^{-}5 = {}^{+}10$$

"If $^{+}2 \cdot {}^{-}5$ suggests the idea of twice a drop of 5° or a drop of 10°, then it seems that the opposite of twice a drop of 5° would be a rise of 10°.

"Finally, we find that we never get into trouble by following this argument. We could express the pattern as:

$$^{-}a \cdot {}^{-}b = {}^{+}ab$$

a	$^{-}1$	$^{+}4$	$^{+}2$									$^{-}3$	
b	$^{+}2$	$^{-}4$		$^{-}5$	$^{+}5$		$^{-}5$						
$a+b$			$^{+}1$	0		$^{+}10$	$^{-}10$	$^{+}15$		$^{+}1$	$^{+}.2$		$^{+}1$
ab					$^{-}25$	$^{+}25$			0	$^{+}\frac{1}{4}$	$^{+}.01$	$^{-}39$	
$a-b$								$^{-}3$	$^{+}18$				$^{+}5$

$r \div m = q$ ____ Division is always possible.

X-Nighyun suggested that he try to answer some of his own questions concerning this claim.

"I wondered what you Earth people might do when faced by

$^{-}6 \div {}^{-}2 =$ ___ or $24 \div {}^{-}3 =$ ___

"My guess is that you would write these expressions as an expression including multiplication with a factor missing. So you might say

$^{-}6 \div {}^{-}2 =$ ___ is the same as ${}^{-}2 \cdot$ ___ $= {}^{-}6$

$24 \div {}^{-}3 =$ ___ is the same as ${}^{-}3 \cdot$ ___ $= 24$

"Now I can see clearly that you would know that

${}^{-}2 \cdot 3 = {}^{-}6$ and ${}^{-}3 \cdot {}^{-}8 = 24$"

"Congratulations. What you've done certainly sounds reasonable to me.

"Now, using any sensible explanation you like, describe situations that fit the idea of the headlines in the next column. Complete the headline first."

$^{+}30 + {}^{-}42 =$ ____

$^{-}7 + {}^{-}6 =$ ____

$^{+}7 \cdot {}^{-}12 =$ ____

$15 \div \frac{2}{3} =$ ____

$^{+}7 + {}^{-}5 =$ ____

x	4	$\frac{1}{2}$		$\frac{1}{2}$	15		19	$^{+}9$				
y		$\frac{1}{8}$										
z	6		9.2			.1						
$x + y$	9			$\frac{5}{6}$		11		$^{+}2$	5	$11\frac{1}{2}$	1.8	$^{+}2$
$y + z$		$\frac{3}{8}$	10.2	$\frac{7}{12}$	110		230		8	$19\frac{1}{2}$	2.5	0
$x + z$			9.9		92	10.1	192	$^{-}4$	7	11	2.3	$^{+}8$

You shouldn't be surprised, one day, if X-Nighyun started asking some strange questions. Like these:

"Now that I understand something about your arithmetic, I'm going to find out something about how clever you are.

"If I made a trip to Astra A from here near your earth's moon, and then returned, it would be 750,000,000 glooms. How far is it from here to Astra A?"

"But how can I tell when I don't know what a gloom is?"

"If you told me that a round trip between New York and Chicago is 1680 miles, I could tell you that those cities are 840 miles apart . . . and I have no idea about the length of your Earth miles."

"Oh, you want to know how many glooms it is from you to Astra A?"

"Of course."

"The distance must be about _____."

"I once made a trip starting from Astra A and going to Astra C, stopping at Astra B on the way. The total trip was 204,000 glooms. It is twice as far from Astra A to Astra B as it is from Astra B to Astra C. How far is it from Astra A to Astra B?"

Answer: _____

"I would have saved 17,500 glooms by going straight from Astra A to Astra C. What is the shortest distance from Astra A to Astra C?"

Answer: _____

"How far is it from here to Astra B if I stop at Astra A on the way?"

Answer: _____

"If my speed is 50,000 glooms per glud, how many gluds will it take for me to get to Astra A from here?"

Answer: _____

"If I returned immediately at half that speed, how long would I have been gone?"

Answer: _____

"On the moon, I weigh 18.5 chorns when I have my space suit on. Now, 40% of that weight is my suit. How much do I weigh?"

Answer: _____

"When I am in my suit and carrying another suit, how much would the total weight be?"

Answer: _____

"A shipment of these space suits weighed 555 chorns. How many space suits were in the shipment?"

Answer: _____

"At 20,000 glooms per glud, how long would it take me to go from Astra C to Astra B?"

Answer: _____

"At the same speed, how long from Astra B to Astra A?"

Answer: _____

"At the same speed, how long from Astra A to my present position near your moon?"

Answer: _____

"Now, it's your turn!"

Here are your questions for X-Nighyun.

1. "Rockton and Newton are 945 miles apart by railroad. A slow freight train — 'S' for short — leaves Rockton and travels 45 miles per hour. How long will it take for S to reach Newton?"

   Answer: _____

2. "A fast freight train — 'F' for short — leaves from Newton and travels at the rate of 60 miles per hour. How long will it take to reach Rockton?"

   Answer: _____

3. "If S left Rockton at 1 P.M., what time did it arrive at Newton?"

   "I suppose it arrived at 22 P.M., whatever 'P.M.' means."

   Always more explaining to do. Could you make X-Nighyun understand our system of talking about time?

4. "All right, I'll change my question. If S left from Rockton and F from Newton at the same time, how much earlier would F reach Rockton than S reached Newton?"

   Answer: _____

5. "How far would S be from Newton when F reached Rockton?"

   Answer: _____

   "S left from Rockton and F left from Newton at 1 P.M. on the same day. Now I have several questions to ask:

6. "How far apart would they be at 2 P.M.?"

   Answer: _____

7. "How far apart at 5 P.M.?"

   Answer: _____

8. "How far apart at 7 P.M.?"

   Answer: _____

9. "How far apart at 1 A.M.?"

   Answer: _____

10. "At what time did they meet?"

    Answer: _____

11. "How far apart at 4 A.M.?"

    Answer: _____

12. "How much further has F to go?"

    Answer: _____

13. "How much further has S to go?"

    Answer: _____

14. "When S reached Junction City, 50% of its fuel had been used up. When it got to Newton, there was still 20% left. About how far is Junction City from Rockton?"

    Answer: _____

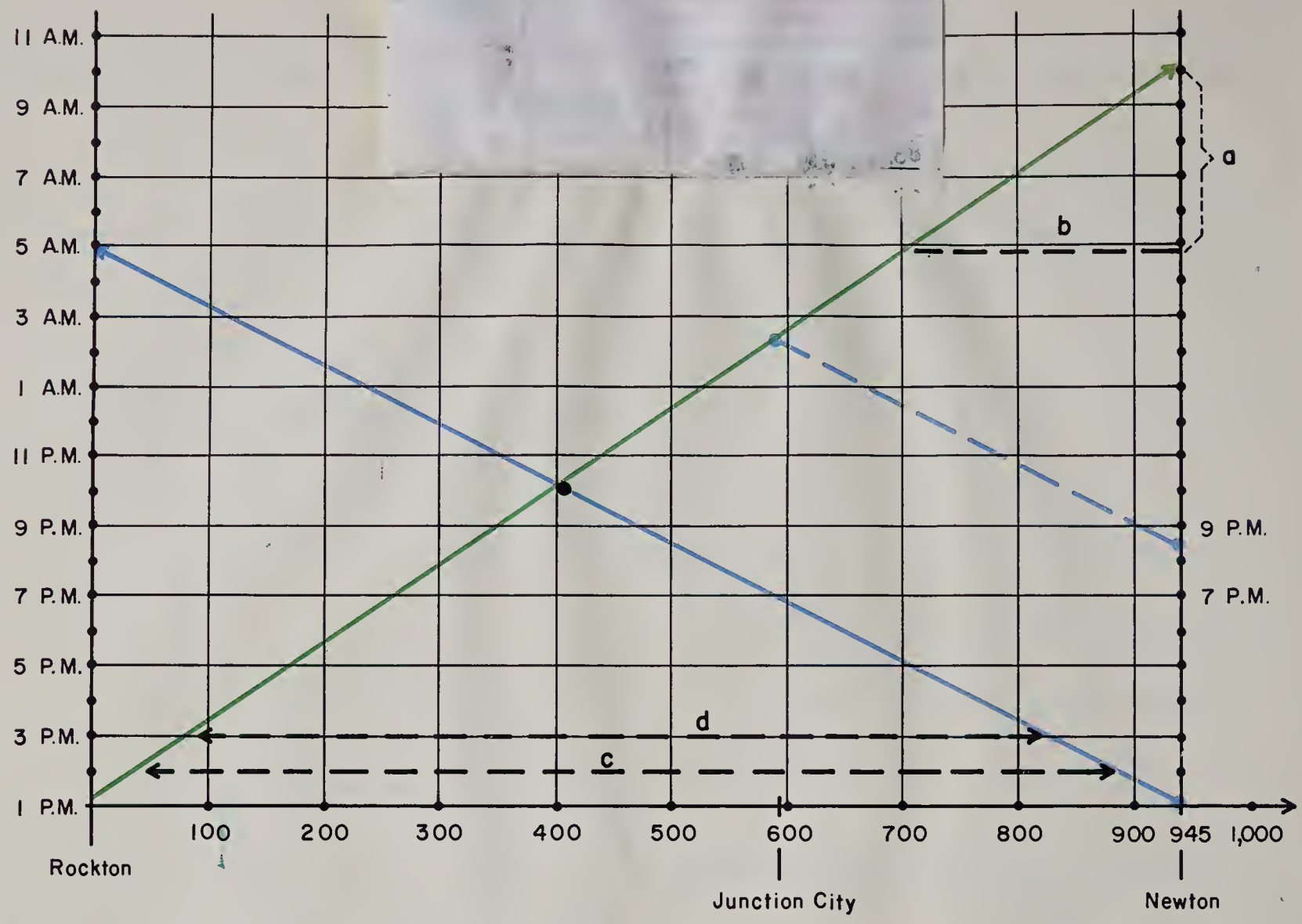
15. "Of course, you could only make a reasonable estimate. Actually, Junction City is 245 miles further from Rockton than from Newton. Now you can tell me how far it is from Junction City to Newton."

    Answer: _____

16. "Now, here is a brainbuster! S left Rockton at 1 P.M. and traveled at the rate of 45 miles per hour. S met F at Junction City; and F had traveled at the rate of 60 miles per hour. At what time did F leave Newton?"

    Answer: _____

    "That sure was a brainbuster. I can work more easily with glooms per glud. How about you?"



By interplanetary mail, X-Nighyun sent a note along with the sketch above.

"While you were asking all those questions about the two freight trains, I used the arithmetic you taught me to find the answers. But I also made a diagram to help me out.

"You see, the diagram contained much of the information you gave and much of the information you made me figure out.

"Another important use for the diagram was that it kept me from making any unreasonable answers. I checked the answer to every problem — except one — by estimating the answer with the diagram above. The exception was that one about fuel.

"I've marked parts of the sketch that I used to help with your third, fourth, and fifth questions.

"I needed to answer the first two questions before I could draw the sketch.

"I'm going to let you decide how I used this diagram to find out whether any answers were reasonable.

"All I did to test my answer to your brainbuster was to draw the broken blue line.

"Thanks for your explanations about arithmetic. I know there is more, but I want to spend some time putting together the pieces you gave me. The patterns help me see what's going on.

"Sincerely,

"X-Nighyun"

Can you discover ways in which X-Nighyun used his sketch?

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